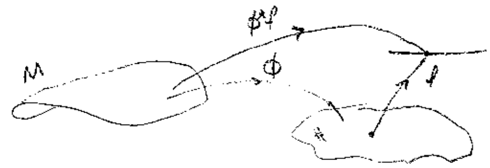
GOLDEN WEDDING OF BLACK HOLES AND THERMODYNAMICS.

BLACK HOLE THERMODYNAMICS MINI-COURSE Lecture I: Preliminary Topics B. Arderucio, 2023

Spacetime: (M, g_{ab}). M diff. manifold, g_{ab} Lorentzian metric.

$\phi: M \to M'$ smooth $f: M' \to \mathbb{R}$ smooth. Pullback $\phi^* f(p) = f(\phi(p))$ defines a map $\phi^* f: M \to \mathbb{R}$



Also, $\begin{cases} \phi_* X(f)\big|_{\phi(p)} = X(\phi^* f)\big|_p & \text{maps } T_p M \text{ to } T_{\phi(p)} M' \\ \phi^*: T^*_{\phi(p)} M' \to T^*_p M \ ; & \phi^* A(X)\big|_p = A(\phi_* X)\big|_{\phi(p)} . \end{cases}$

$\to$ ϕ_* and ϕ^* can be extended to arbitrary (r,s) tensors by passing all the composed s vectors and r points as arguments.

Let ϕ_t be a one-parameter group of diffeomorphisms ($\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$) Let X denote the vector field such as $\phi_t(p)$ are its integral curves. $\mathcal{L}_X T\big|_p = \lim_{t\to 0} \frac{1}{t}\left(T\big|_p - \phi_{t\,*} T\big|_p \right)$ (Lie derivative)

Exponential map $\exp: T_p M \to M$ $\exp X = q$, q is reached from p by following the geodesic from p with tangent X one unit in its affine parameter

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$I^+(p) = \{ q \in M \mid \exists$ timelike curve $\gamma(t)$ with $\gamma(0) = p$ and $\gamma(1) = q \}$ $\quad I^+(S) = \bigcup_{p\in S} I^+(p)$. (chronological future)

$J^+(p) = \{ q \in M \mid \exists$ non-spacelike curve $\gamma(t)$ with $\gamma(0) = p$ and $\gamma(1) = q \} \cup \{p\}$. $\quad J^+(S) = \bigcup_{p\in S} J^+(p)$ (causal future)

$E^+(S) = J^+(S) \setminus I^+(S)$ (horismos)

$D^+(S) = \{ q \in M \mid \forall$ past-inextendible non-spacelike curve through p intersects $S \}$ (future Cauchy development)

$\overline{D^+(S)} \setminus I^-(D^+(S)) = H^+(S)$ (future Cauchy horizon)

S is said to be achronal if $I^+(S) \cap S = \emptyset$.

If S is achronal $\text{edge}(S) = \{ q \in \bar{S} \mid \exists \mathcal{O} \ni q \ \exists p \in I^-(q, \mathcal{O}) \ \exists r \in I^+(q, \mathcal{O})$ so that there is a timelike curve in $\mathcal{O}$ that does not intercept $S \}$.

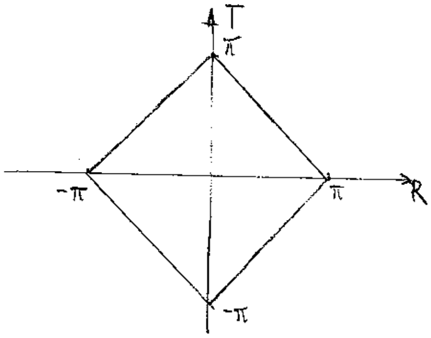
Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$ $\quad ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ $\quad u = t - r, \quad v = t + r$

$$ds^2 = -du\,dv + \frac{1}{4}(v-u)^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (u, v \in \mathbb{R})$$

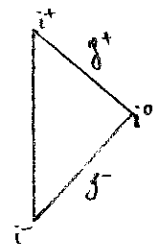
$u = \tan U, \; v = \tan V \quad , \quad U, V \in (-\pi/2, \pi/2) \qquad ds^2 = \sec^2 U \sec^2 V \left[-dU\,dV + \frac{1}{4}\sin^2(V-U)(d\theta^2 + \sin^2\theta\, d\phi^2) \right]$

Consider $g_{ab} = \Omega^2(x)\,\eta_{ab}$ with $\Omega^2(x) = 4\cos^2 U \cos^2 V > 0 \;\; \forall U, V$. The line element $d\tilde{s}^2$ for g_{ab} is

$$d\tilde{s}^2 = -4\,dU\,dV + \sin^2(V-U)(d\theta^2 + \sin^2\theta\, d\phi^2) = -dT^2 + dR^2 + \sin^2 R\,(d\theta^2 + \sin^2\theta\, d\phi^2) \text{ for } T = U + V,\; R = V - U$$



g_{ab} and η_{ab} share the same null geodesics



Carter-Penrose conformal diagram

$$r = \frac{1}{2}\left[\tan\frac{T+R}{2} - \tan\frac{T-R}{2}\right]$$

$\Rightarrow V \geq U$

We append to M a boundary, ∂M. Use it to define asymptotic flatness.

Def: Black hole: $B = M \setminus J^-(\mathcal{J}^+, M \cup \partial M)$. Event Horizon: $\dot{B} = \dot{J}^-(\mathcal{J}^+, M \cup \partial M)$

For a collapsing null shell: $ds^2 = -\left(1 - \frac{2m\,\theta(v)}{r}\right)dv^2 + 2\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ $\qquad (\rho \propto M'(v).)$

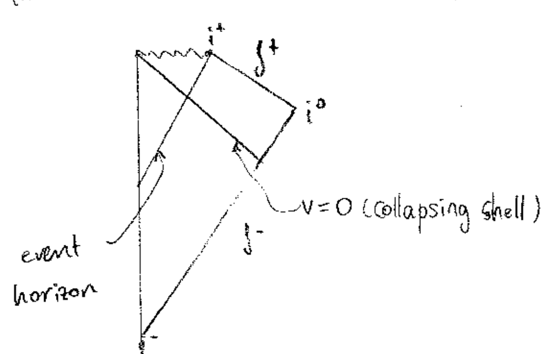
Define the retarded coordinate $\; u = v - 2r \quad v < 0 \qquad ds^2 = -du\,dv + \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

For $v > 0$, the outgoing null curves solve $\left(1 - \frac{2m}{r}\right)dv = 2\,dr \Rightarrow v = 2\left(r + 2m\log\left|\frac{r}{2m} - 1\right|\right) + f(u)$ $\quad (r > 2m)$

Continuity across $v = 0$ demands $f(-2r) + 2\left(r + 2m\log\left(\frac{r}{2m} - 1\right)\right) = 0 \Rightarrow f(u) = -2\left(-\frac{u}{2} + 2m\log\left(\frac{-u}{4m} - 1\right)\right)$

The outgoing rays are $\; v - u + 2m\log\left(-\frac{u}{4m} - 1\right) = 2\left(r + 2m\log\left(\frac{r}{2m} - 1\right)\right)$ $\qquad (u < -4m)$

Now as $u \to -4m$ the l.h.s $\to -\infty$. $\quad u = -4m$ locates the event horizon



BLACK HOLE THERMODYNAMICS MINI-COURSE   Lecture 2: Zeroth Law   B. Arderucio, 2023

In GR, gravity is not a force $\Rightarrow$ observers may enjoy following a timelike isometry and fail to be inertial. Explicitly, let (M, g_{ab}) be stationary with Killing field ξ^a. An observer following the orbits of ξ^a has $u^a = \xi^a / \sqrt{-\xi^b \xi_b}$

$N^2 = -\xi^c \xi_c$. Their acceleration is

$$\frac{\xi^a}{N} \nabla_a \left(\frac{\xi^b}{N} \right) = \frac{1}{N^2} \xi_a \nabla^a \xi^b + \frac{1}{N} \xi_a \xi^b \left(\frac{1}{2} \frac{2 \xi_c \nabla^a \xi^c}{N^3} \right) = \underbrace{\frac{1}{N^2} \xi_a \nabla^a \xi^b}_{= \frac{1}{N} \nabla^b N} + \underbrace{\frac{1}{N^4} \xi^b \xi_a \xi_c \nabla^a \xi^c}_{= 0, \text{ Killing eq.}}$$

$$\left[\nabla^b N = \nabla^b (-\xi^c \xi_c)^{\frac{1}{2}} = -\frac{1}{2} (-\xi^c \xi_c)^{-1/2} (2) \xi_a \nabla^b \xi^a = -\frac{1}{N} \xi_a \nabla^b \xi^a = +\frac{1}{N} \xi_a \nabla^a \xi^b \right]$$

Non-constant N is a signature of gravity. The corresponding force is $F = m \frac{1}{N} \nabla^b N = m \nabla^b \log N$

If (M, g_{ab}) is asymptotically flat, the force exerted at infinity is $(\nabla_b E \nabla^b E)^{1/2}$ for $E = -m \xi^a \frac{\xi_a}{N}$

$$F_\infty = m \left(\nabla_b \left(\frac{\xi^a \xi_a}{N} \right) \nabla^b \left(\frac{\xi^c \xi_c}{N} \right) \right)^{1/2} = m \left(\nabla_b N \nabla^b N \right)^{1/2} = N F.$$

In a point where $N \to 0^+$ $F \to \infty$, but maybe not F_∞. Let's use this!

First, $R_{abcd} \xi^d = \nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c \doteq \nabla_a \nabla_b \xi_c + \nabla_b \nabla_c \xi_a$. Analogously

$$R_{bcad} \xi^d = \nabla_b \nabla_c \xi_a + \nabla_c \nabla_a \xi_b \quad +$$

$$R_{cabd} \xi^d = \nabla_c \nabla_a \xi_b + \nabla_a \nabla_b \xi_c \quad -$$

$R_{[abc]d} = 0$

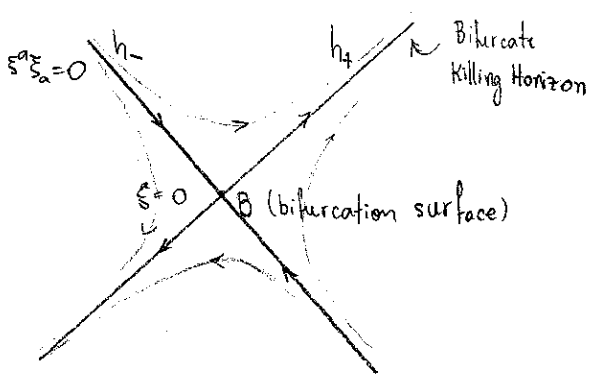
$$-R_{bca}{}^d \xi_d = \nabla_a \nabla_b \xi_c \qquad (I)$$

If we know $\xi^a(p)$ and $\nabla^b \xi^a(p)$, we can use (I) to evolve the field ξ^a.

Suppose the Killing field vanishes on a $(n-2)$-dimensional spacelike surface B and that $\nabla^a \xi^b |_B \neq 0$.

For $p \in B$, the action of the isometry generated by ξ^a on $T_pM \ni A^a$ is $\mathcal{L}_\xi A^a = [\xi, A]^a = -A^a \nabla_a \xi_b = \nabla_b \xi_a A^a$

$\nabla_b \xi_a$ is anti-symmetric. Then, just like for the Lorentz group in Minkowski spacetime,



On h_- and h_+ $\xi^a \xi_a = 0$. Then

(II) $\nabla^a (\xi^b \xi_b) = -2\kappa \xi^a$ for a function κ. $\Rightarrow$

$$\left. \begin{array}{l} \Rightarrow \underbrace{\mathcal{L}_\xi (\nabla^a \xi^b \xi_b)} = -2 \xi^a (\mathcal{L}_\xi \kappa) \\ \xi^c \nabla_c \nabla^a (\xi^b \xi_b) - \nabla^c \xi^b \xi_b \nabla_c \xi^a = 0 \end{array} \right\} \Rightarrow \boxed{\mathcal{L}_\xi \kappa = 0}$$

From (II), $\xi^b \nabla_b \xi^a = \kappa \xi^a$ (III) $\begin{cases} \text{Contract with } \ell^a \text{ (null) such that } \xi^a \ell_a = -1 \text{ to obtain } \kappa = -\xi^a \ell^b \nabla_b \xi_a \\ \text{Multiply by } \ell^c \text{ and anti-symmetrise: } -2\ell^{[b}\xi^{c]} \nabla_c \xi^a = \kappa \epsilon^{ab} \end{cases}$

$$-2\kappa^2 = \kappa \epsilon^{ab} \kappa \epsilon_{ab} = 4\left(\ell^{[b}\xi^{c]} \nabla_c \xi^a\right)\left(\ell_{[b}\xi_{d]} \nabla^d \xi_a\right) = 4\ell^{[b}\xi^{c]} \ell_{[b}\xi_{d]} \nabla_c \xi^a \nabla^d \xi_a = \frac{4}{4}\left(\xi^c \ell_d + \ell^c \xi_d\right)(\nabla_c \xi^a)(\nabla^d \xi_a) =$$

$$= \left(\xi^c \ell_d + \ell^c \xi_d - m^c \bar{m}_d - \bar{m}^c m_d\right) \nabla_c \xi^a \nabla^d \xi_a = g^c{}_d \nabla_c \xi^a \nabla^d \xi_a \Rightarrow \left(\nabla^a \xi^b \nabla_a \xi_b\right) = -2\kappa^2 \quad \text{(IV)}$$

Taking a derivative of (IV) along a direction t^a tangent to B, $2\kappa t^a \nabla_a \kappa = -t^c (\nabla_c \nabla_a \xi_b)(\nabla^a \xi^b) \overset{(I)}{=}$

$= t^c R_{abc}{}^d \xi_d \nabla^a \xi^b = 0$ (on B). When $\kappa \neq 0$, this yields $\boxed{t^a \nabla_a \kappa = 0}$

Also, (III) is the geodesic equation for a non-affine parameter: $\frac{D^2 x^a}{d\sigma^2} = \kappa(\sigma) \frac{Dx^a}{d\sigma}$.

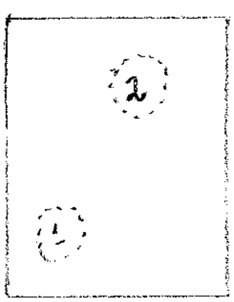
$$\frac{D^2 x^a}{d\tau^2} = \frac{D}{d\tau}\left(\frac{Dx^a}{d\sigma}\frac{d\sigma}{d\tau}\right) = \frac{d^2\sigma}{d\tau^2}\frac{Dx^a}{d\sigma} + \left(\frac{d\sigma}{d\tau}\right)^2 \underbrace{\frac{D^2 x^a}{d\sigma^2}}_{=\kappa \frac{Dx^a}{d\sigma}} \qquad \text{Choose } \tau;\ \frac{d^2\sigma}{d\tau^2} = -\kappa(\sigma)\left(\frac{d\sigma}{d\tau}\right)^2 \Rightarrow$$

$$\Rightarrow \kappa = -\frac{d^2\sigma/d\tau^2}{(d\sigma/d\tau)^2} = \frac{-\frac{d}{d\tau}\log\frac{d\sigma}{d\tau}}{d\sigma/d\tau} = \frac{d}{d\tau}\log\frac{d\tau}{d\sigma} \Rightarrow \frac{d\tau}{d\sigma} = \exp\int^\sigma \kappa(s)\, ds \quad \text{and} \quad \frac{D^2 x^a}{d\tau^2} = 0.$$

$$\frac{E_\infty}{m} = \left(-\xi^d \xi_d\right)^{1/2} \left[\frac{\xi^b \nabla_b \xi^c}{-\xi^a \xi_a} \frac{\xi^e \nabla_e \xi_c}{-\xi^f \xi_f}\right]^{1/2} = \left(-\xi^d \xi_d\right)^{-1/2} \left[\kappa^2 \xi^c \xi_c\right]^{1/2} = |\kappa| \quad \text{(interpretation)}$$

Near horizon geometry: $ds^2 = -\kappa^2 r^2 dt^2 + dr^2 + 2g_{u\alpha}\, du\, dx^\alpha + g_{\alpha\beta} dx^\alpha dx^\beta \qquad \xi = \frac{\partial}{\partial u}, \quad t = u + \int \frac{d\rho}{-\xi^a \xi_a}$

A derivation similar to the Unruh effect, $T = \frac{\kappa}{2\pi\sqrt{-\xi^a \xi_a}}$ $\qquad (\rho = 0 \text{ at } b_\pm) \qquad r = \sqrt{2\rho/\kappa}$



$E_0 := -g(p, \xi)$

$E_{1,2} = -g(p, u_{1,2}) = -g\left(p, \frac{\xi}{\sqrt{-g(\xi,\xi)}}\Big|_{1,2}\right) = \frac{E_0}{\sqrt{-g(\xi,\xi)}}$

energy leaves 1 and arrives at 2.

For extensive systems, $dS_1 + dS_2 = 0$. Using $\frac{d}{dE_1} = -\frac{\sqrt{-g(\xi,\xi)}|_1}{\sqrt{-g(\xi,\xi)}|_2}\frac{d}{dE_2}$ and $T_{1,2} = \frac{dS_{1,2}}{dE_{1,2}}$

$T_1 \sqrt{-g(\xi,\xi)}|_1 = T_2 \sqrt{-g(\xi,\xi)}|_2$ (Tolman relation).