Black Hole Thermodynamics Lecture Notes: First Law of Black Hole Mechanics

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1 Introduction

The goal of this lecture is to derive the first law of black hole mechanics. This is a very important equation in black hole thermodynamics as it resembles almost exactly the first law of thermodynamics. The first law of thermodynamics is written

$$dU = đQ + đW \tag{1.1}$$

where U is the internal energy, Q is heat flow, W is work done on the system. đ represents an inexact differential[1]. This can be written as

$$dU = TdS + pdV \tag{1.2}$$

plus any other work terms that may be relevant.

Our first law of black hole mechanics is going to look like this:

$$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ_H \tag{1.3}$$

where M is the mass of a black hole, κ is the surface gravity, A is the area of the event horizon, Ω_H is the angular velocity and J_H is the angular momentum. Note that throughout this lecture I assume black holes are uncharged. All the results we derive can be extended to charged black holes.

The first law of black hole mechanics is obviously formally analogous to the first law of black hole thermodynamics, but it is also somewhat physically analogous. The internal energy of a black hole really is just its mass, and the rotational energy of a black hole is a measure of the work it can do (as I discuss more in lecture 6). Of course, the analogy breaks down for the entropy term as a classical black hole has zero temperature. It isn't until the

[1] I.e. a differential that is path dependent.

discovery of Hawking radiation that the area term can also be interpreted as physically analogous to entropy term. But we are getting ahead of ourselves. Our job today is to derive the first law of black hole mechanics.

We will do this in four parts. First, in order to have quantities like the mass, area of angular momentum of a black hole, we need to integrate over spacetime regions. Therefore, our first job is to learn how to do integration on manifolds. Once we can integrate, we have to find quantities that can describe the mass and angular momentum of a black hole. These will be the Komar integrals, and describe this quantities for stationary black holes.

Once we have defined M and J_H, we can derive the *integral form* of the first law of black hole mechanics. This is a relationship between the absolute quantity of the mass, angular momentum and area of a black hole.

Finally, we derive a differential form of the first law, which describes the relationship between infinitesimal changes in these quantities.

2 Integration in General Relativity

Let M be a differentiable manifold. A p-form on M is an anti-symmetric $(0, p)$-tensor on M.

The exterior derivative of a p-form, α, is defined as

$$[d\alpha]_{a_1 \ldots a_{p+1}} = (p+1)\nabla_{[a_1}\alpha_{a_2 \ldots a_{p+1}]} \tag{2.1}$$

A manifold of dimension n is orientable if it admits an orientation: a smooth, nowhere vanishing $n - form$, $\boldsymbol{\epsilon}_{a_1 \ldots a_n}$. Two orientations are equivalent is $\boldsymbol{\epsilon}' = f\boldsymbol{\epsilon}$ where f is an everywhere positive function. A orientation is right handed in a given coordinate chart if it is everywhere positive, and left handed if it is negative.

Note: Any n-form X is related to $\boldsymbol{\epsilon}$ by $X = h\boldsymbol{\epsilon}$ for some function h. X will define an orientation provided h does not vanish, hence an orientable manifold admits precisely two inequivalent orientations.

On an oriented manifold with a metric, the volume form is defined by

$$\boldsymbol{\epsilon}_{12 \ldots n} = \sqrt{|g|} \tag{2.2}$$

and the contraction of two volume forms for a n-dimensional Lorentzian spacetime is given by

$$\boldsymbol{\epsilon}^{a_1 \ldots a_j a_{j+1} \ldots a_n}\boldsymbol{\epsilon}_{a_1 \ldots a_j b_{j+1} \ldots b_n} = -(n-j)!j!\delta^{[a_{j+1}}_{b_{j+1}} \ldots \delta^{a_n]}_{b_n}, \tag{2.3}$$

Let M be an oriented manifold of dimension n. Let $\psi : \mathcal{O} \to \mathcal{U}$ be a right handed coordinate chart, $\{x^\mu\}$ and let X be a n-form. The integral of X *over* $\mathcal{O}$ is

$$\int_{\mathcal{O}} X \equiv \int_{\mathcal{U}} dx^1 \ldots dx^n X_{12 \ldots n} \tag{2.4}$$

which is chart independent. To extend this over M, we simply sum over the integration of all charts in an atlas, weighted by a partition of unity.

The volume of M, and the integral of a function f on M, are given respectively by

$$\int_M \boldsymbol{\epsilon}, \qquad \int_M f\boldsymbol{\epsilon} \tag{2.5}$$

A manifold with boundary is defined in the same way as a manifold except that charts map to open subsets of $\frac{1}{2}\mathbb{R}^n = \{(x^1...x^n) \in \mathbb{R}^n | x^1 \geq 0\}$. This is just a technical way of way some dimension of the manifold is cut off at a boundary and has the natural interpretation.

Stokes' theorem in general relativity:

Theorem 2.1. *Stokes' Theorem*

Let M be an n-dimensional compact oriented manifold with boundary and let $\boldsymbol{\alpha}$ be an (n-1)-form on M which is C^1. Then

$$\int_M d\boldsymbol{\alpha} = \int_{\dot{M}} \boldsymbol{\alpha} \tag{2.6}$$

where the dot denotes boundary, and d is the exterior derivative.

3 Komar Integrals

Now that we can integrate, we want an expression for the mass energy of a black hole spacetime (or any spacetime for that matter).

In general relativity, the stress-energy tensor T_{ab} represents the energy properties of matter. Using this the local energy properties relative to any observer will be well-defined. $\nabla^a T_{ab} = 0$ represents local energy conservation, as it is defined at every point, but it does not in general lead to a global conservation law. However, if the spacetime admits a timelike Killing vector field then we do have a global conservation law in the usual form:

Lemma 3.1. *If a spacelike admits a timelike Killing vector field ξ^a, then for any two spacelike hypersurfaces Σ, Σ' which bound a region R, the total energy of matter on Σ is equal to the total energy of matter on Σ' where the total energy of matter is given by*

$$E(\Sigma) = -\int_\Sigma \boldsymbol{\epsilon}_{a_1a_2a_3b} T^b_c \xi^c \tag{3.1}$$

I.e. E is a conserved quantity.

Proof.

$$\nabla^a(T_{ab}\xi^b) = (\nabla^a T_{ab})\xi^b + T_{ab}\nabla^a \xi^b = 0 \tag{3.2}$$

Define

$$E(\Sigma) = -\int_\Sigma \boldsymbol{\epsilon}_{a_1a_2a_3b} T^b_c \xi^c \tag{3.3}$$

where $\boldsymbol{\epsilon}_{a_1a_2a_3b}$ is the volume form on the spacetime.

Now let Σ and Σ' bound a spacetime region R, as in figure 1. By Stokes' theorem

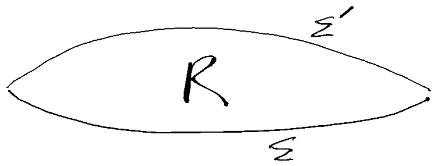

Figure 1. Kerr Black Hole Conformal Diagram

$$E(\Sigma') - E(\Sigma) = -\int_{\dot{R}} \boldsymbol{\epsilon}_{a_1 a_2 a_3 b} T^b_c \xi^c \tag{3.4}$$

$$= -4 \int_R \nabla_{[d} \boldsymbol{\epsilon}_{a_1 a_2 a_3] b} T^b_c \xi^c \tag{3.5}$$

$$= -4 \int_R \boldsymbol{\epsilon}_{[a_1 a_2 a_3 | b} \nabla_{|d]} T^b_c \xi^c \tag{3.6}$$

Any n-form must be proportional to $\boldsymbol{\epsilon}$. So

$$\boldsymbol{\epsilon}_{[a_1 a_2 a_3 | b} \nabla_{|d]} T^b_c \xi^c = h \boldsymbol{\epsilon}_{a_1 a_2 a_3 d} \tag{3.7}$$

contracting the LHS and the RHS with $\boldsymbol{\epsilon}^{a_1 a_2 a_3 d}$ gives

$$\nabla_b T^b_c \xi^c = h \tag{3.8}$$

where we have used 2.3.

The LHS of (3.7) is zero, and so the RHS is also zero. Thus,

$$E(\Sigma') - E(\Sigma) = 0 \tag{3.9}$$

□

Unfortunately, this quantity $E(\Sigma)$ is no use. At no point have we used the Einstein equation. T_{ab} therefore cannot be describing the total energy of the spacetime in any useful sense as it cannot be capturing the energy possessed by, for example, a black hole in a vacuum spacetime. Indeed, T_{ab} need not be the tensor that appears on the RHS of Einstein's equation for the above derivation to be valid. We are clearly missing something.

Unfortunately again, there is no meaningful notion of the energy density of the gravitational field in general relativity, so we cannot state a local conservation law for gravitational energy and follow the same procedure as above. But fortunately, there does exists a useful notion of the total energy of an isolated system, where by isolated we mean it can be modelled as asymptotically flat.

The main desiderata of any adequate account of mass is that if a spacetime contains mass M, a particle at $r >> M$ should behave as if acted on by a Newtonian mass M. What we are going to do is model the force required to keep a shell of unit mass density stationary around the region of spacetime that contains all the gravitational energy. We will find this expression has the same form as the Newtonian expression for a shell around a mass M. Therefore we equate the expressions, and thus get an expression for the mass. Before modelling this, let's prove a lemma that we will need:

Lemma 3.2.

$$\nabla_{[a}(\boldsymbol{\epsilon}_{bc]de}\nabla^d\xi^e) = \frac{2}{3}R^e_d\xi^d\boldsymbol{\epsilon}_{eabc} \tag{3.10}$$

Proof.

$$\begin{aligned}\boldsymbol{\epsilon}^{abcd}\nabla_b(\boldsymbol{\epsilon}_{cdef}\nabla^e\xi^f) &= \boldsymbol{\epsilon}^{abcd}\boldsymbol{\epsilon}_{cdef}\nabla_b\nabla^e\xi^f \\ &= -4\nabla_b\nabla^{[a}\xi^{b]} \\ &= 4\nabla_b\nabla^b\xi^a\end{aligned} \tag{3.11}$$

(where the second line uses (2.3) and the third line uses the Killing property.)

Using the definition the Killing property and the Ricci identity $\nabla_a\nabla_b w^c + \nabla_b\nabla_a w^c = R^c_{dab}w^d$, where w^a is any vector field, one can prove

$$\nabla_a\nabla_b\xi_c = -R^d_{bca}\xi_d \tag{3.12}$$

which implies

$$\nabla_a\nabla^a\xi_b = -R^c_b\xi_c \tag{3.13}$$

thus (3.11) gives

$$\boldsymbol{\epsilon}^{abcd}\nabla_b(\boldsymbol{\epsilon}_{cdef}\nabla^e\xi^f) = -4R^a_b\xi^b \tag{3.14}$$

Contracting both sides with $\boldsymbol{\epsilon}_{almn}$ gives

$$\nabla_{[a}(\boldsymbol{\epsilon}_{bc]de}\nabla^d\xi^e) = \frac{2}{3}R^e_d\xi^d\boldsymbol{\epsilon}_{eabc} \tag{3.15}$$

as required. □

Now, let us model our stationary shell of mass surrounding the region full of energy. Consider a static, asymptotically flat spacetime that is vacuum near infinity. Normalise the Killing vector field ξ^a so that $|\xi| = (-\xi_a\xi^a)^{1/2} \to 1$ at infinity. The four-velocity of a particle following an orbit of ξ^a (i.e. stationary) is $v^a = \xi^a/|\xi|$, and the four-acceleration is

$$a^b = v^a\nabla_a v^b \tag{3.16}$$

which is the local force that must be exerted to hold a unit mass in place. The force that must be exerted at infinity is red-shifted by a factor $|\xi|$ to give $\tilde{a}^b = v^a\nabla_a\xi^b$. Thus, suppose we have a shell S of unit surface mass density in the Cauchy surface orthogonal to v^a. Then the force required to keep this shell on a stationary orbit is just the integral of $\tilde{a}^b$ normal to S over the surface S.

$$F = \int_S n^b v^a \nabla_a \xi_b \boldsymbol{\epsilon}_{cd} \tag{3.17}$$

where n^a is the unit vector normal to S, and $\boldsymbol{\epsilon}_{cd}$ is the volume element on S.

By Killing's equation we can throw away the symmetric part of $n^b v^a$, leaving $v^{[a}n^{b]}$. Now $N^{ab} = 2v^{[a}n^{b]}$ is the unit tensor normal to S. Thus $-6N_{[ab}\boldsymbol{\epsilon}_{cd]} = \boldsymbol{\epsilon}_{abcd}$, the volume form on the entire spacetime. Substituting this in

$$F = -\frac{1}{2}\int_S \boldsymbol{\epsilon}_{abcd}\nabla^c \xi^d \tag{3.18}$$

Now we need to show this is independent of S. Let S be entirely in the vacuum region, so $R_{ab} = 0$. Then by our lemma 3.1, $\nabla_{[a}(\boldsymbol{\epsilon}_{bc]de}\nabla^d\xi^e) = 0$. Applying Stokes' theorem to any volume V bounded by two surfaces S and S' in the vacuum region, one gets

$$\int_V \nabla_{[a}(\boldsymbol{\epsilon}_{bc]de}\nabla^d\xi^e) = \int_S \boldsymbol{\epsilon}_{bcde}\nabla^d\xi^e - \int_{S'} \boldsymbol{\epsilon}_{bcde}\nabla^d\xi^e = 0 \tag{3.19}$$

And so our force F is independent of S.

In Newtonian gravity the force required to hold a shell of unit mass density in place is also independent of the shape of the shell and is given by $4\pi M$. We therefore equate this force in general relativity and write

$$M = \frac{-1}{8\pi}\int_S \boldsymbol{\epsilon}_{abcd}\nabla^c\xi^d \tag{3.20}$$

This is called the Komar mass, and is the mass of all stationary asymptotically flat spacetimes that are vacuum near infinity. This is obviously conserved, and it is clearly very closely associated with the time translation symmetry; a nice feature of a definition of mass indeed. We also have used Einstein's equation in this derivation, making sure therefore it is fundamentally general relativistic.

I state without proof that one can also write the angular momentum of an axisymmetric spacetime as

$$J = \frac{1}{16\pi}\int_S \boldsymbol{\epsilon}_{abcd}\nabla^c\psi^d \tag{3.21}$$

where ψ^a is the spacelike Killing vector field which generates a 1-parameter group of isometries isomorphic to $U(1)$. J here is the angular momentum of the spacetime, and both J and M are equal to their notational equivalents in the Kerr spacetime.

The Komar mass is only for stationary spacetimes. For non-stationary spacetimes, we use the ADM mass. For this we restrict attention to those spacetimes that admit a 3 + 1 decomposition, which allows us to write down a Hamiltonian formulation of general relativity. One finds that the Hamiltonian takes the form $H = H_0 + H'$ where $H_0 = 0$ for any solution satisfying the constraint equations, and H' a surface term. Therefore, $E_{ADM} = H'$, which takes the form of an integral over the boundary of a 3-slice.

Armed with well defined notions of mass, let's derive the integral form of the first law of black hole mechanics.

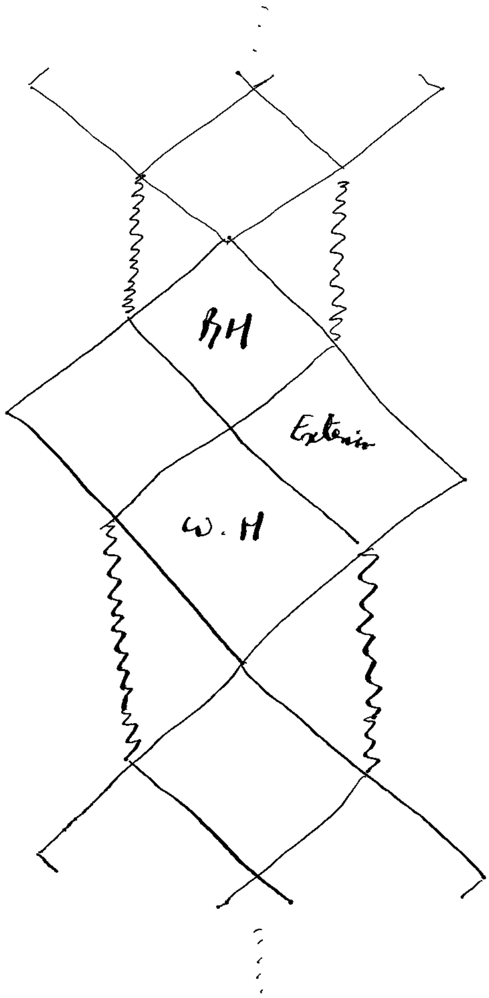

Figure 2. Kerr Black Hole Conformal Diagram

4 Integral Form

Consider a Kerr Black Hole, the unique solution for an uncharged, spherically symmetric spacetime. The conformal diagram for such a black hole is given by figure 2. However, we believe that physical black holes form from the collapse of stellar matter. Modelling this matter as uncharged, we replace most of our conformal diagram with this matter, which gives us a new diagram depicted in figure 3. The final mass and angular momentum of the black hole is determined by the mass and angular momentum of the collapsed matter.

Now, consider a Cauchy surface that intersects the horizon after the matter has all crossed the horizon. Such a Cauchy surface, call it Σ, is displayed in figure 3. Let $\mathcal{H}$ be the event horizon, and let H be the 2-surface $\mathcal{H} \cap \Sigma$. Thus, H is a boundary to Σ.

We know that the Kerr black hole admits a timelike killing vector field ξ^a, and azimuthal killing vector field, ψ^a, and another killing vector field that is tangent to the generators of the event horizon on the horizon

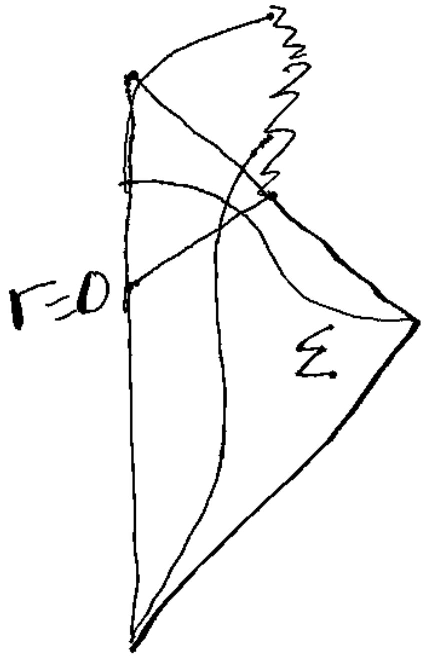

Figure 3. Kerr Black Hole Conformal Diagram

$$\chi^a = \xi^a + \Omega_H \psi^a \tag{4.1}$$

where Ω_H is interpreted as the angular velocity of the black hole.[2] These will be used to calculate the mass and angular momentum of our black hole.

From the previous section, the Komar integral for the total mass of a stationary, asymptotically flat spacetime is given by

$$M = \frac{-1}{8\pi} \int_S \boldsymbol{\epsilon}_{abcd} \nabla^c \xi^d \tag{4.2}$$

where S is a two sphere at infinity.

We want an expression for the M in terms of state variables. To get this, we apply Stokes' theorem to our expression for the Komar mass.

$$\begin{aligned} M &= \frac{-1}{8\pi} \int_S \boldsymbol{\epsilon}_{abcd} \nabla^c \xi^d \\ &= \frac{-1}{8\pi} \int_\Sigma d(\boldsymbol{\epsilon}_{abcd} \nabla^c \xi^d) \end{aligned} \tag{4.3}$$

[2]This interpretation comes from modelling the angular velocity $d\phi/dt$ of an observer who's angular velocity comes entirely from the rotation of the spacetime. As such an observer approaches the event horizon of a rotating black hole, the angular velocity becomes Ω_H. See [1] chapter 12.3 for details.

We can split this integral into two parts; one over the interior of the black hole to H, and the other from H to infinity. Call these portions Σ_{int} and Σ_{ext} respectively. For the integral over Σ_{int} we can use Stokes' theorem again to turn the integral over the interior of the black hole into an integral on H.

$$M = \frac{-1}{8\pi}\int_{\Sigma_{ext}} d(\boldsymbol{\epsilon}_{abcd}\nabla^c\xi^d) - \frac{1}{8\pi}\int_H \boldsymbol{\epsilon}_{abcd}\nabla^c\xi^d \tag{4.4}$$

Regarding the first term

$$\begin{aligned}
&\frac{-1}{8\pi}\int_{\Sigma_{ext}} d(\boldsymbol{\epsilon}_{abcd}\nabla^c\xi^d)\\
&= \frac{-3}{8\pi}\int_{\Sigma_{ext}} \nabla_{[a}(\boldsymbol{\epsilon}_{bc]de}\nabla^d\xi^e)\\
&= \frac{-1}{4\pi}\int_{\Sigma_{ext}} R^e_d\xi^d\boldsymbol{\epsilon}_{eabc}\\
&= -2\int_{\Sigma_{ext}} (T^e_d - \frac{1}{2}T\delta^e_d)\xi^d\boldsymbol{\epsilon}_{eabc}
\end{aligned} \tag{4.5}$$

where in the third line we have used (3.10), and the in the fourth line we have used Einstein's equation.

For the second term, we use equation (4.1) so that

$$\begin{aligned}
&\frac{1}{8\pi}\int_H \boldsymbol{\epsilon}_{abcd}\nabla^c\xi^d\\
&= \frac{1}{8\pi}\int_H \boldsymbol{\epsilon}_{abcd}\nabla^c\chi^d - \Omega_H\frac{1}{8\pi}\int_H \boldsymbol{\epsilon}_{abcd}\nabla^c\psi^d\\
&= \frac{1}{8\pi}\int_H \boldsymbol{\epsilon}_{abcd}\nabla^c\chi^d - 2\Omega_H J_H
\end{aligned} \tag{4.6}$$

where we have used the (3.21), the Komar integral for the angular momentum J_H of the black hole.

To evaluate the first term in the last line of (4.6), note that the volume element $\boldsymbol{\epsilon}_{ab}$ on H is given by $\boldsymbol{\epsilon}_{ab} = \boldsymbol{\epsilon}_{abcd}n^c\chi^d$ where n^a is tangent to the future directed null normal to H, normalised to $n^a\chi_a = -1$. Thus

$$\boldsymbol{\epsilon}^{ab}\boldsymbol{\epsilon}_{abcd}\nabla^c\chi^d = \boldsymbol{\epsilon}^{abef}n_e\chi_f\boldsymbol{\epsilon}_{abcd}\nabla^c\chi^d = -4n_c\chi_d\nabla^c\chi^d = -4\kappa \tag{4.7}$$

where for the second equality we have used again (2.3), and for the third we have used the definition of surface gravity, $\chi^b\nabla_a\chi_b = -\kappa\chi_a$. Therefore, we can write

$$\int_H \boldsymbol{\epsilon}_{abcd}\nabla^c\chi^d = \frac{1}{2}\int_H (\boldsymbol{\epsilon}^{ef}\boldsymbol{\epsilon}_{efcd}\nabla^c\chi^d)\boldsymbol{\epsilon}_{ab} = -2\kappa A \tag{4.8}$$

Hence,

Theorem 4.1. *First Law of Black Hole Mechanics for an Uncharged Black Hole (Integral Form)*

$$M = 2\int_{\Sigma_{ext}} (T^e_d - \frac{1}{2}T\delta^e_d)\xi^d \boldsymbol{\epsilon}_{eabc} + \frac{1}{4\pi}\kappa A + 2\Omega_H J_H \tag{4.9}$$

The first term can be viewed as a contribution from the matter in the region exterior to the black hole, and the second and third terms can be viewed as the contribution by the black hole. Therefore, we can write down a vacuum version of the first law in integral form

Corollary 4.1.1. *First Law of Black Hole Mechanics in Vacua for an Uncharged Black Hole (Integral Form)*

$$M = \frac{1}{4\pi}\kappa A + 2\Omega_H J_H \tag{4.10}$$

This is a useful expression but holds little philosophical interest. It is just an equation of state for the black hole. Indeed, the first law of thermodynamics is only well stated in its differential form; there is no good notion of the absolute amount of work possessed by any thermodynamic system. Therefore, if we want to draw an analogy between black hole mechanics and thermodynamics, we need to write the above first law in a differential form. I will describe two ways to this result.

5 Differential Form

5.1 Energy Balance

To calculate how the variables change with respect to each other, we need to consider two spacetimes and calculate the different Komar integrals in each. Thus we consider two spacetimes (M, g) and $(M, g + \delta g)$. We define

$$h^{ab} = \delta g^{ab} = -g^{ac}g^{bd}\delta g_{cd} \tag{5.1}$$

as the perturbation metric. Consider this to be a tensor field in the spacetime (M, g). Now define the vector

$$v^a = \nabla_b(h^{ab} - g^{ab}h) \tag{5.2}$$

where $h = h^a_a$.

Note that $\nabla_a v^a = 0$ is the trace of the perturbed Einstein equation ($-\frac{1}{2}\nabla_a\nabla_c h - \frac{1}{2}\nabla^b\nabla_b h_{ac} + \nabla^b\nabla_{(c}h_{a)b} = 0$), and so is true by Einstein's equation (see chapter 7 in [1]). We also have $\nabla_a\xi^a = 0$ by Killing's equation. Also note that for a stationary perturbation, $\mathcal{L}_\xi v^a = 0$.[3] The Lie derivative of a vector field Y^a with respect to X^a is equal to the commutator of these vector fields, $[X, Y]^a$. Thus, $[\xi, v]^a = 0$.

Given these properties we have

[3]As a reminder, the Lie derivative $\mathcal{L}_X T$ is found by comparing the tensor T at a point p to the push forward of T by the diffeomorphism that maps points to points a parameter distance t along the integral curves of X, where X is a vector field, in the limit as $t \to 0$.

Lemma 5.1.

$$[d(\boldsymbol{\epsilon}_{bcde}v^d\xi^e)]_a = 0 \tag{5.3}$$

Proof.

$$\nabla_a(v^{[a}\xi^{b]}) = \frac{1}{2}(v^a\nabla_a(\xi^b) - \xi^a\nabla_a(v^b)) = 0 \tag{5.4}$$

where for the first equality we have used $\nabla_a\xi^a = \nabla_a v^a = 0$, and for the second we have used $[\xi, v]^a = 0$.

Therefore,

$$[d(\boldsymbol{\epsilon}_{bcde}v^d\xi^e)]_a = 3\nabla_{[a}(\boldsymbol{\epsilon}_{bc]de}v^d\xi^e) = 0 \tag{5.5}$$

where the second equality follows from (5.4).

□

Therefore, by Stokes' theorem, if we take Σ to be some three-dimensional volume bounded by two two-spheres, $\dot{\Sigma}_1$ and $\dot{\Sigma}_2$, we find

$$\int_\Sigma d(\boldsymbol{\epsilon}_{bcde}v^d\xi^e) = 0 = \int_{\dot{\Sigma}_1}\boldsymbol{\epsilon}_{bcde}v^d\xi^e - \int_{\dot{\Sigma}_2}\boldsymbol{\epsilon}_{bcde}v^d\xi^e \tag{5.6}$$

Now let $\dot{\Sigma}_1 = S$, the two sphere at infinity, and $\dot{\Sigma}_2 = H$, the intersection of the horizon with a Cauchy surface, we get,

$$\int_S \boldsymbol{\epsilon}_{bcde}\xi^e\nabla_f(h^{fd} - g^{fd}h) = \int_H \boldsymbol{\epsilon}_{bcde}\xi^e\nabla_f(h^{fd} - g^{fd}h) \tag{5.7}$$

It is proven by Bardeen, Carter and Hawking (1973) in [2] that the LHS is equal to $8\pi dM$, and that the RHS is equal to $-2Ad\kappa - 16\pi J_H d\Omega_H$, thus

$$dM = \frac{1}{8\pi}(-2Ad\kappa - 16\pi J_H d\Omega_H) \tag{5.8}$$

Returning to equation (4.10), we vary this to get

$$dM = \frac{1}{4\pi}(Ad\kappa + \kappa dA) + 2(J_H d\Omega_H + \Omega dJ_H) \tag{5.9}$$

and we add equations 5.8 and 5.9 together to get

Theorem 5.2. *First Law of Black Hole Mechanics for an Uncharged Black Hole (Differential Form)*

$$dM = \frac{1}{8\pi}\kappa dA + \Omega dJ_H \tag{5.10}$$

In reality, we have modelled the the variation between two spacetimes, (M, g) and $(M, g + \delta g)$, where the δg represents a slight increase in the mass of the black hole, and we track the changes to the angular momentum and area on the horizon. This approach can be generalized to non-stationary perturbations, non-vacuum spacetimes and also to include a electric potential term for charged black holes. We now turn to another approach.

5.2 Partial Derivative Method

This method relies on another result: the so-called 'no-hair theorem'. The heuristic statement of the theorem is that a black-hole is completely described by three parameters, its mass M, its charge Q and its angular momentum J. However, the 'no-hair theorem' is a misnomer, because there are in fact many theorems that contribute to the impression that black holes are completely described by these three parameters, and none of these theorems proves exactly what is stated in the heuristic version. I will therefore call this the 'no-hair conjecture'.

Consider as an example a foundational theorem which contributes to the no-hair conjecture is the following:

Theorem 5.3. *Kerr Uniqueness 1 [3] [4]*

If (M, g) is a stationary, axisymmetric, asymptotically flat vacuum spacetime suitably regular on, and outside, a connected event horizon then (M, g) is a member of the 2-parameter Kerr [5] family of solutions. The parameters are mass m and angular momentum J, and $|J| < M$.

The limitation of theorems such as this is clear; it only applies to vacuum spacetimes, and thus clearly do not guarantee the conjecture holds for astrophysical black holes. Results such as these [6] have been generalised somewhat [7], but still a completely general proof of the conjecture is not available.[4] Despite this, support for the conjecture is very strong, and for this derivation we will assume the conjecture is true.

Given the conjecture, we can write

$$A|_{Q=0} = A(M, J) \tag{5.11}$$

Going forwards, I will suppress the $Q = 0$ subscript, keeping the uncharged assumption implicit. Assuming that this function is C^1 in both variables, we can then write

$$dA(M, J) = \left.\frac{\partial A}{\partial M}\right|_J dM + \left.\frac{\partial A}{\partial J}\right|_M dJ \tag{5.12}$$

We can't use expression (4.10) to compute these partial derivatives as this is also a function of κ and Ω_H. Instead, we use results from the analysis of the Kerr metric. In Boyer-Lindquist coordinates, (t, r, θ, ϕ), the Kerr metric takes the form

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2(\theta)}{\Sigma}dt^2 - 2a\sin^2(\theta)\frac{r^2 + a^2 - \Delta}{\Sigma}dtd\phi \\ &+ \left(\frac{(r^2 + a^2)^2 - \Delta a^2\sin^2(\theta))}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \end{aligned} \tag{5.13}$$

where $a = J_H/M$

[4] Indeed, Haco et al [8] have argued that black holes have 'soft-hair', but these results are beyond the scope of the current discussion.

$$\Sigma = r^2 + a^2 \cos^2\theta \tag{5.14}$$

$$\Delta = r^2 - 2Mr + a^2 \tag{5.15}$$

There is a coordinate singularity at $\Delta = 0$, which when solved gives $r_\pm = M \pm \sqrt{M^2 - a^2}$. These correspond to the inner and outer horizons, which have interesting properties we will not study here. We will only be concerned with the outer horizon, which is the radius at which null geodesics cannot escape. We want to know the area of H, defined by r_+. The induced metric on H for a Cauchy surface such that $dt = 0$ will be $h = g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2$. The volume form on this orientable two-surface is then $\boldsymbol{\epsilon}_{\theta\phi} = \sqrt{|h|}$. Hence we can calculate the area of H.

$$\begin{aligned} A &= \int_{r=r_+} \sqrt{g_{\theta\theta}g_{\phi\phi}}d\theta d\phi \\ &= \int_{r=r_+} (r_+^2 + a^2) sin\theta d\theta d\phi \\ &= 4\pi(r_+^2 + a^2) \\ &= 4\pi(M^2 + 2M\sqrt{M^2 - a^2} + M^2 - a^2 + a^2) \\ &= 8\pi(M^2 + M\sqrt{M^2 - a^2}) \end{aligned} \tag{5.16}$$

I claim without proof that, from the definitions of κ and Ω_H, for the Kerr metric

$$\kappa = \frac{\sqrt{M^2 - a^2}}{2M^2 + 2M\sqrt{M^2 - a^2}} \tag{5.17}$$

$$\Omega_H = \frac{a}{2M^2 + 2M\sqrt{M^2 - a^2}} \tag{5.18}$$

Therefore, computing our partial derivatives gives

$$\left.\frac{\partial A}{\partial M}\right|_J = \frac{8\pi}{\kappa} \tag{5.19}$$

and

$$\left.\frac{\partial A}{\partial J}\right|_M = \frac{-8\pi\Omega_H}{\kappa} \tag{5.20}$$

Thus we recover theorem 5.2, the first law of black hole mechanics in differential form.

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