GOLDEN WEDDING OF BLACK HOLES AND THERMODYNAMICS

BLACK HOLE THERMODYNAMICS MINI-COURSE   Lecture 4. Area Theorem Part I   B. Arderucio, 2023

$Z^a = (\partial/\partial t)^a$

$\frac{D}{dv}V^a = 0$ $\lambda(t,v)$ for every fixed t is the point obtained from $\lambda(t)$ after moving an affine parameter v along the integral curves of V^a. ($\mathcal{L}_V Z^a = 0$)

congruence of geodesics.

$$\Downarrow$$

$$\frac{D}{dv}Z^a = \frac{D}{dt}V^a \quad \text{(I)}$$

v and $f(t)v$ are valid affine parameters for null geodesics

The co-dimension 1 subspace orthogonal to V^a includes V^a itself because V^a is null. We quotient this subspace identifying vectors that only differ from each other by a multiple of V^a. Denote this co-dimension 2 subspace by S_pM. If L^a is the other null vector with $g(V,L) = -1$, $h_{ab} = g_{ab} + 2V_{(a}L_{b)}$ projects a vector onto $(S_pM)_\perp$. For null surfaces, it induces a Riemannian metric. From (I), $\frac{D}{dv}Z^a = Z^b\nabla_b V^a$. Hence

$$\frac{D^2 Z^a}{dv^2} = V^e\nabla_e(V^c\nabla_c Z^a) = V^e\nabla_e(Z^c\nabla_c V^a) = V^e\nabla_e Z^c\nabla_c V^a + V^e Z^c\nabla_e\nabla_c V^a = V^e\nabla_e Z^c\nabla_c V^a + V^e Z^c\nabla_c\nabla_e V^a$$

$$- R_{ced}{}^a V^d V^e Z^c = Z^e\nabla_e V^c\nabla_c V^a + V^e Z^c\nabla_c\nabla_e V^a - R_{ced}{}^a V^d V^e Z^c = Z^e\nabla_e(\overset{0}{\cancel{V^c\nabla_c V^a}}) - R_{ced}{}^a V^d V^e Z^c.$$

$$\Rightarrow \frac{D^2 Z^a}{dv^2} = -R^a{}_{bcd} Z^c V^b V^d \quad \text{(Jacobi eq. / Geodesic Deviation eq.)} \quad \text{(II)}$$

$\nabla_c(V^a L_a) = 0$ $V^b\nabla_b(Z^a V_a) = V_a V^b\nabla_b Z^a = V_a Z^b\nabla_b V^a = 0$. So we take Z^a such that $g(Z,L) = 0|_p$.

The solutions to $\frac{D}{dv}Z^a = \nabla_b V^a Z^b$ (1st order linear ODE) are of the form $Z^a(v) = E^a{}_b(v)\, Z^b|_q$.

Plugging into (I) and (II), $\frac{d}{dv}E_{ab} = \nabla_c V_a E^c{}_b$ (III) and $\frac{d^2}{dv^2}E_{ab} = -R_{acde}V^c V^e E^c{}_b$ (IV)

<u>Def.</u>: $\omega_{ab} = h_a{}^c h_b{}^d \nabla_{[d}V_{c]}$ (vorticity tensor) $\theta_{ab} = h^c{}_a h^d{}_b \nabla_{(d}V_{c)}$

$\sigma_{ab} = \theta_{ab} - \frac{1}{2}h_{ab}\theta$ (shear tensor) $\theta = \nabla^a V_a$ (expansion)

$\frac{d}{dv}\det E = \det(\theta_{ca}E^c{}_b) = \det\theta_{ac}\det E$

$$h^a{}_c h^d{}_b \nabla^c V^d = \frac{1}{2}\theta h_{ab} + \sigma_{ab} + \omega_{ab}$$

From (III), $\omega_{ab} = -(E^{-1})_{c[a}\frac{d}{dv}E_{b]}{}^c$

$\theta_{ab} = (E^{-1})_{c(a}\frac{d}{dv}E_{b)}{}^c$ $\mathrm{Tr}\left(\frac{dA}{dx}A^{-1}\right) = \frac{1}{\det A}\frac{d}{dx}\det A$

$\theta = \frac{1}{\det E}\frac{d}{dv}\det E$

From (IV), $\frac{d^2}{dv^2}E_{ab} = -R_{acde}V^c V^e E^c{}_b \Rightarrow$

$\Rightarrow \frac{d}{dv}\omega_{ab} = 2\omega_{c[a}\theta_{b]}{}^c \Rightarrow$

$\Rightarrow \frac{d}{dv}(E_{ca}\omega^c{}_d E^d{}_b) = 0 \Rightarrow$

From the above: → Because $\frac{d}{dv}\det E$ is not singular everywhere (otherwise (IV) would break), $\theta \to -\infty \Leftrightarrow \det E \to 0$

→ $E_{ca}\,\omega^c{}_d\,E^d{}_b$ is a constant along each geodesic. In particular, if one starts with a point q in which $E=0$, $\omega_{ab}=0$ wherever E is regular.

Def.: A point $p \in \gamma(v)$ is conjugate to $q \in \gamma(v)$ if $\exists\, J^a \in \mathfrak{X}(M)$ nonzero, Jacobi field that vanishes at p and q.

Multiplying (IV) by $E^{-1}{}_{bc}$ and taking the symmetric part; $\frac{d}{dv}\theta_{ab} = -R_{acbd}V^cV^d - \omega_{ac}\omega^c{}_b - \theta_{ac}\theta^c{}_b$, whose trace yields $\frac{d}{dv}\theta = -R_{ab}V^aV^b + 2\omega^2 - 2\sigma^2 - \frac{1}{2}\theta^2$ (Landau-Raychaudhuri eq.).

If one seeks conjugate points of q, $2\omega^2 = 0$ and $\frac{d}{dv}\theta = -R_{ab}V^aV^b - 2\sigma^2 - \frac{1}{2}\theta^2$.

If $R_{ab}V^aV^b \geq 0$ everywhere (see below) $\frac{d\theta}{dv} \leq -\frac{1}{2}\theta^2 < 0 \Rightarrow \theta \leq \dfrac{2}{v - \left(v_0 - \frac{2}{\theta(v_0)}\right)}$

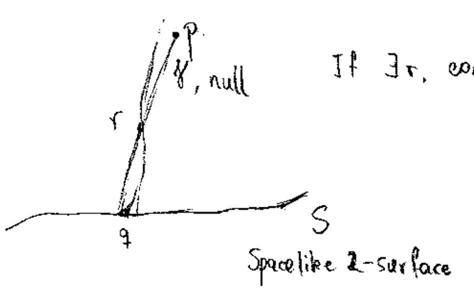
and there will be a conjugate point if $\theta(v_0) < 0$ (unless the geodesic can't be extended this far).

Null energy condition (NEC) states that $T_{ab}V^aV^b \geq 0$ for all null vectors V^a. If one uses Einstein's Eq,

$R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}g_{ab}T^c{}_c\right)$, this is equivalent to $R_{ab}V^aV^b \geq 0$.

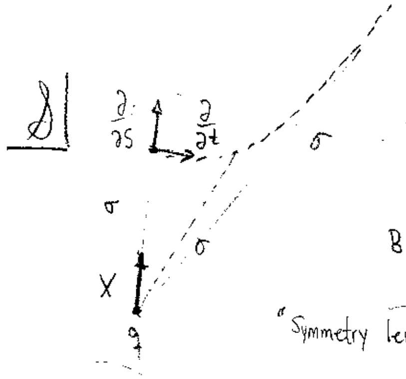
The NEC follows from the Weak Energy Condition (WEC).

For timelike geodesics, focusing requires $R_{ab}W^aW^b \geq 0$ for all timelike W^a. This is the Strong Energy Condition.



If ∃ r, conjugate point, we can deform γ to create a timelike curve from S to p.

We'll discuss this in the next lecture.



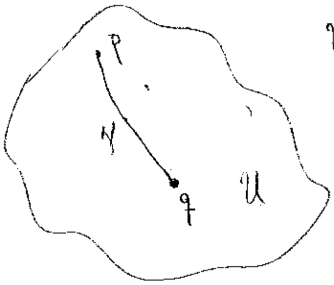
$$\frac{\partial}{\partial s} g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = \underbrace{g\left(\frac{D}{\partial s}\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)}_{0} + g\left(\frac{\partial}{\partial s}, \frac{D}{\partial s}\frac{\partial}{\partial t}\right) \Rightarrow \frac{\partial}{\partial s} g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = \frac{1}{2}\frac{\partial}{\partial t} g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = 0$$

But $\frac{D}{\partial s}\frac{\partial}{\partial t} = \mathcal{L}_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} + \frac{D}{\partial t}\frac{\partial}{\partial s} = \frac{D}{\partial t}\frac{\partial}{\partial s}$ on γ

"Symmetry lemma".

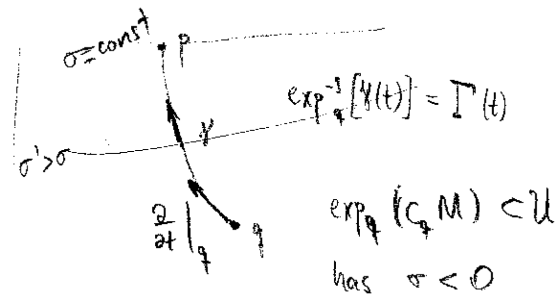
$g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)$ is constant in s. Because $\frac{\partial}{\partial t}\Big|_q = 0$ and $q \in \gamma$, $g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)\Big|_\gamma = 0$. Hence,

the timelike geodesics through q are orthogonal to the surfaces $\{p \in M \mid g(\exp_q^{-1} p, \exp_q^{-1} p) = \sigma\}$.



U convex normal neighbourhood of q.

$\exists X \in T_p M$; $p = \exp_q X$



Since $\frac{\partial}{\partial t}$ is orthogonal to $\sigma = \text{const}$, $p \in \exp_q(C_q M)$.

If is not timelike, but still causal, take $Y \in T_q M$; $\exp_{q*}(Y)$ is timelike with $g(Y, \frac{\partial}{\partial t})\Big|_q < 0$

Reapeat the analysis for the integral curve of $\frac{\partial}{\partial t} + \epsilon Y$ ← for each ϵ. Conclude that p is reached by $\overline{\exp_q(C_q M)} = \exp_q(\overline{C_q M})$.

→ If $p \in U$ can be reached by a causal curve from q that is not timelike, p lies on a null geodesic from q.

→ As a corollary, $I^{\pm}(p)$ are open sets.