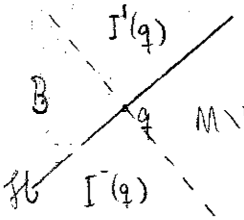
BLACK HOLE THERMODYNAMICS MINI-COURSE Lecture 5. Area Theorem Part 2 B Arderucio, 2023

1. The event horizon is an achronal set, i.e., $I^+(\mathcal{H}) \cap \mathcal{H} = \phi$

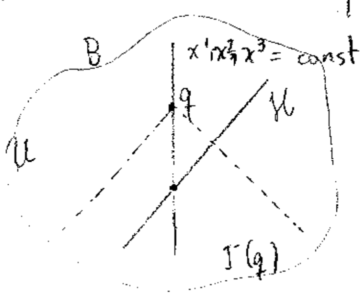
<u>Proof.</u> We first show that for $q \in \mathcal{H}$, $I^-(q) \subset M \setminus B$. If $p \in I^-(q)$ $\exists \mathcal{V}$, neighbourhood of q $\mathcal{V} \subset I^+(p)$ [because $I^+(p)$ is open]. But $q \in \dot{B} \Rightarrow \mathcal{V}$ intercepts $M \setminus B$. Then $p \in I^-(\mathcal{V} \cap (M \setminus B)) \subset M \setminus B$

Analogously, $I^+(q) \subset B$. Now to the main result. Suppose $\exists r \in \mathcal{H};\ r \in I^-(q)$. Because $I^-(q)$ is open $\exists \mathcal{V}' \subset I^-(q) \subset I^-(\mathcal{H})$ with $r \in \mathcal{V}'$, but that is impossible because $I^-(\mathcal{H})$ is open and cannot intercept its boundary.



We can endow $\mathcal{H}$ with manifold structure (imbedded submanifold) by introducing Riemann normal coordinates (x^0, x^1, x^2, x^3) around $q \in B$, with x^0 timelike. In the neighbourhood where they are defined, the curves for constant x^1, x^2, x^3 must intercept both $I^+(q)$ and $I^-(q)$ and contain exactly one point of $\mathcal{H}$ (because $\mathcal{H}$ is achronal). The value of x^0 at the intersection obeys $|x^0(q_1) - x^0(q_2)|^2 \leq \sum_{i=1}^{3} (x^i(q_1) - x^i(q_2))^2$ $q_1, q_2 \in \mathcal{H}$ (since q_1 and q_2 are not timelike separated). We use $\phi: B \cap \mathcal{U} \to \mathbb{R}^3$, associating the point q with its integral curve $\phi(q) = (x^1(q), x^2(q), x^3(q))$ defines local chart. An atlas is built by reapeating this procedure around other points in B.

↓ for $\mathcal{H} \cap \mathcal{U}$



ϕ is a homeomorphism in the induced topology

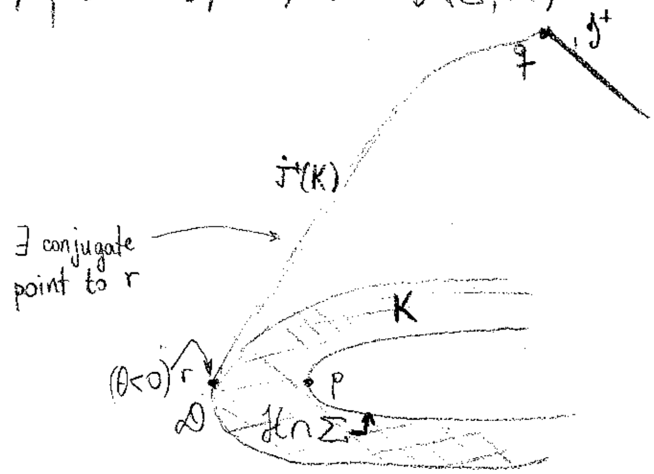
x^0 is a Lipschitz continuous (constant 1) of $x^{1,2,3}$

Let's sketch a proof that every point on $\mathcal{H}$ lies on a future-inextendible null geodesic with no future endpoints

Choose a sequence of points p_n on $I^-(\mathscr{I}^+)$ that converges to $p \in \mathcal{H}$. From the definition of $\mathscr{I}^+$ each p_n is connected to $\mathscr{I}^+$ by a curve λ_n, which is future-inextendible

Construct a 'limit curve'. If there were a point of λ in $I^-(\mathscr{J}^+)$, λ itself would connect p to the exterior, which is impossible ($\mathcal{B}$ is closed). So $\lambda \subset \mathcal{H}$.

Let's assume that (M, g) is future asymptotically predictable from Σ, i.e., $\mathscr{J}^+ \subset D^+(\Sigma, \overline{M})$ (no 'naked' singularities in the future of Σ).



But, from last class, the existence of this conjugate point implies that $\exists$ a timelike curve connecting this pair of points, a contradiction.

→ $\theta \geq 0$ everywhere on $\mathcal{H}$ for a NEC-obeying asymptotically future predictable spacetime.

The area of $\mathcal{H} \cap \Sigma_2$ is at least the area of $\mathcal{H} \cap \Sigma_1$ (S. Hawking, 1971).