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Quantum Field Theory in Curved Spacetime

An Introduction

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“Have you guessed the riddle yet?” the Hatter said, turning to Alice again.

“No, I give it up,” Alice replied. “What’s the answer?”

“I haven’t the slightest idea,” said the Hatter.

“Nor I,” said the March Hare.

A piece of a conversation between Alice, the Mad Hatter and the March Hare, in Lewis Carroll’s *Alice’s Adventures in Wonderland*.

Abstract

This is an introduction to quantum field theory in curved spacetimes written for the minicourse presented at the Golden Wedding of Black Holes and Thermodynamics: An Online Celebration. It includes discussions about the algebraic approach, the Fock space approach, the path integral approach, and particle detectors suitable for someone with previous exposure to non-relativistic quantum mechanics and special relativity. Knowledge of general relativity and quantum field theory in flat spacetime is recommended, but not mandatory. The content follows closely, and sometimes overlaps with, the author's master's thesis, [1].

Keywords: quantum field theory in curved spacetime, algebraic quantum field theory, path integrals, particle detectors, Unruh effect.

Abbreviations

CCR canonical commutation relations

GNS Gelfand–Naimark–Segal

GR general relativity

KMS Kubo–Martin–Schwinger

LHS left-hand side

QFT quantum field theory

QFTCS quantum field theory in curved spacetime

RHS right-hand side

SM standard model

UV ultraviolet

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One

Why Do Things Fall?

We begin by discussing why quantum field theory in curved spacetime is an interesting subject and what are its relevancy and limitations. In addition, we clearly specify the general problems we want to solve with QFTCS. This is a preparation so that the following chapters can answer these questions.

1.1 Silly Big Questions

Some questions in physics are so deep and profound that they could be asked by a toddler. One could call them "silly big questions". Some examples are "how did the universe begin?", "what are things made of?", "can we go back in time?", "what is time?" and, of course, "why do things fall?".

Some of these questions have been asked at least since the ancient Greeks [46]. Aristotle, for example, believed all things were made of water, earth, fire, air, and aether. The motion of everything, including gravity, was explained in terms of these five elements and their natural properties. While simple, such an explanation is surprisingly good and was the apex of human science for nearly two millennia. Nevertheless, the theory did have flaws and has been updated over the last few centuries. A modern-day physicist would answer those two questions in the following manners. Everything is made from the quantum fields described by the standard model (SM) [47, 62, 64]; and things fall because spacetime is curved according to the Einstein equations of general relativity (GR) [28, 59].

The modern description does lack an interesting feature of the Aristotelian theory, though. While Aristotle's theory has a unified description of what things are made of and why they fall, modern theories can barely fit together in a clear manner. More specifically, GR—our best description of how things fall—is a theory written with classical matter in mind. Nevertheless, the SM—the best description of what things are made of—is a quantum theory, which describes matter in a manner that is not the same considered in GR. Hence, it is necessary to make at least some conceptual modifications to the theories to make them fit together. Notice this is not the more complicated problem of quantum gravity, which aims at understanding how one can get a fully quantum theory of gravitational phenomena. Instead, the point is merely that GR is a classical theory, and hence to consider quantum matter one must somehow modify it.

GR can be roughly understood according to a famous phrase due to John A. Wheeler [65]: spacetime tells matter how to move and matter tells spacetime how to curve (see Fig. 1.1). In order to consider the presence of quantized matter, we will update this description to also allow spacetime to tell quantum fields how to evolve, but we will make a simplifying compromise of not letting the quantum fields affect spacetime. While this would be necessary for a more detailed analysis, we will ignore it to keep our model simple. The resulting framework is then known as quantum field theory in curved spacetime (QFTCS), as pictured in Fig. 1.2.

In which regimes is this an interesting description? Whenever the classical spacetime curvature is already so large that we can treat the quantum fields as being "very light", *i.e.*, such that they do not affect the curvature very much. This can be assumed to be true in contexts in which the field's state is assumed to be "not too energetic", in these sense that its energy-momentum tensor can be assumed to have a small expectation value and small fluctuations. This is the case, for example, in particle physics, when one assumes the fields themselves do not curve spacetime strongly enough to spoil the approximation of a flat spacetime. Similarly, we will work in conditions in which the fields to not spoil the background spacetime, with the difference that

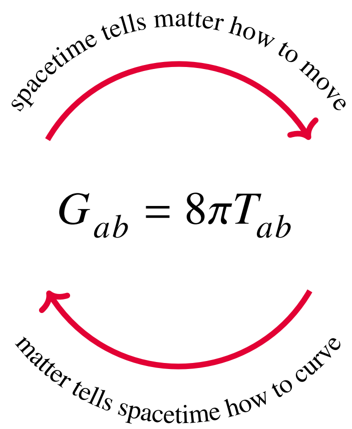

Figure 1.1: Illustration of how GR works. On the left-hand side one has the Einstein tensor G_{ab}, which is a geometrical quantity describing the curvature of spacetime. On the right-hand side one has the stress-energy-momentum tensor T_{ab}, which models the matter content on spacetime. This expression, the Einstein field equations, is the main equation behind GR and describes how matter and spacetime influence each other.

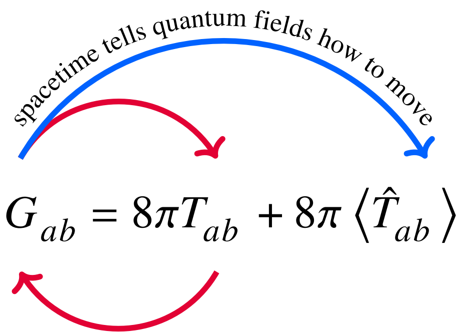

Figure 1.2: Illustration of how QFTCS works. On the right-hand side, in addition to the classical stress-energy-momentum tensor, we now also have a quantum stress tensor representing the quantum matter on spacetime. Within QFTCS, we allow the spacetime geometry to dictate the dynamics of these quantum fields, but make the simplifying assumption that they do not impact the background geometry in a significant way. While this is, of course, a simplification, it still allows us to obtain interesting results.

we shall consider more general backgrounds and frames of reference. For example, QFTCS is generous in accelerated reference frames and near black holes, which shall be our main interests throughout this text.

One should point out, however, that while Einstein said "subtle is the Lord, but malicious he is not", generous are the Einstein equations, but malicious too they are. Many interesting solutions to the EFEs are interesting, but present unusual causal relations. For example, they might exhibit time-travel or faster-than-light travel features [48], or completely miss a physical interpretation [28, Sec. 5.8]. Hence, we should begin by restricting our attention to a certain class of spacetimes of interest.

1.2 Spacetimes of Interest

To keep our domain of interest limited to physically relevant spacetimes, we shall demand some good causal properties of the classical background. For example, we demand the classical theory we want to quantize to already be well-defined. We do this by requiring the existence of a structure known as a Cauchy surface.

Definition 1.1 [Cauchy Surface]**:**
Let (M, g_{ab}) be a spacetime. We say $\Sigma \subseteq M$ is a *Cauchy surface* if, and only if, Σ is such that

i. Σ is topologically closed;

ii. all inextendible timelike curves intersect Σ exactly once;

iii. all inextendible causal curves intersect Σ at least once. ♠

This definition is equivalent to the one given in [59], where more details can be found.

The interest in a Cauchy surface lies on the fact that it ensures the whole spacetime can be characterized by a single hypersurface. All events in the spacetime are affected by or affect the Cauchy surface. Furthermore, given an event p to the future of the Cauchy surface, the everything to the past of p eventually crossed the Cauchy surface (and similarly for events at the past of the hypersurface). Hence, giving complete information about a classical field on the Cauchy surface allows one to obtain complete information about the field on the entire spacetime. This is put in precise mathematical form, for example, in the review [21].

The existence of a Cauchy surface is not guaranteed to all spacetimes. Anti-De Sitter (AdS) spacetime, for example, does not have a Cauchy surface. Neither do any spacetimes with closed timelike curves (which thus allow time-travel). Our focus will be on those spacetimes which do possess a Cauchy surface. These are said to be globally hyperbolic.

Definition 1.2 [Globally Hyperbolic Spacetimes]**:**
We say a spacetime is globally hyperbolic if, and only if, it possesses a Cauchy surface. ♠

Within these spacetimes, we are ensured to be able to solve, for example, the Klein–Gordon equation [21],

$$(\nabla_a \nabla^a - m^2)\varphi = j, \tag{1.2.1}$$

where ∇_a is the Levi-Civita connection associated with the metric g_{ab}, and j is an arbitrary source.

Let us then quantize the Klein–Gordon equation. To do so, however, we shall wonder what *is* a quantum theory.

1.3 What is a Quantum Theory?

To begin answering what a quantum theory *is*, we shall first focus on what a general physical theory is, or should be. To do so, we shall follow [1, 3] and references therein.

Consider any physical experiment. In order to be able to predict its outcomes, it is reasonable to expect that the results are somehow ruled by an underlying probability distribution. Given an apparatus prepared in some specific manner A, a physical system prepared in some specific manner ω, and some arbitrary result q, we expect that there is a probability $p(q|\omega, A)$ such that

$$p(q|\omega, A) = \lim_{N\to\infty} \frac{N_q}{N}, \tag{1.3.1}$$

where N is the amount of times the experiment has been realized under the same conditions (A and ω) and N_q is the amount of times the result turned out to be q. If this structure is not possible, one can hardly imagine how to predict the outcomes of this experiment, and hence we are no longer dealing with a physics experiment.

Quite often, it is possible that using different apparatuses or slightly different preparations of the same apparatus will lead to the same probabilistic outcomes. If it happens that

$$p(q|\omega, A_1) = p(q|\omega, A_2) \tag{1.3.2}$$

for all q's and for all ω's, we shall say that A_1 and A_2 correspond to the same *observable*. The collection of all observables is denoted by $\mathcal{A}$.

Similarly, it is possible that two different copies of a system or two different preparations of the same system lead to the same results for any observable. Namely,

$$p(q|\omega_1, A) = p(q|\omega_2, A), \tag{1.3.3}$$

for all q and for all $A \in \mathcal{A}$. In this case, we say ω_1 and ω_2 correspond to the same *state*.

The outcomes q can in principle be anything. They are usually an image in a digital display or a position in some sort of analogue ruler attached to the apparatus (such as the marks on an ammeter). In any case, it is convenient to map these results to the real line, so it is easier to mathematically manipulate them. From here onward, we will assume the outcomes of physical experiments are always expressed as real numbers, so that $q \in \mathbb{R}$.

This allows us to build the notion of a function of an observable. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $A \in \mathcal{A}$. Then we define $f(A)$ to be the observable that yields $f(q)$ whenever A would yield q. This allows us, for example, to define A^n as the observable that yields q^n whenever A would yield q.

Using this, we can write simply the moments of the probability distribution $p(q|\omega, A)$. Namely, we define

$$\omega(A) = \int q p(q|\omega, A) \,\mathrm{d}q\,, \tag{1.3.4}$$

and notice that

$$\omega(A^n) = \int q^n p(q|\omega, A) \,\mathrm{d}q\,. \tag{1.3.5}$$

With this information, we can reconstruct the information about the probability distribution we actually are trying to describe. Performing this reconstruction is known as the Hausdorff moment problem [49]. A simpler route is to pick the function

$$\chi_r(q) \equiv \begin{cases} 1, & \text{if } q = r, \\ 0, & \text{otherwise,} \end{cases} \tag{1.3.6}$$

and notice that $p(r|\omega, A) = \omega(\chi_r(A))$. Therefore, knowing $p(q|\omega, A)$ or knowing $\omega(A)$ are just two different "coordinate systems" for knowing the state ω.

We shall now focus on figuring out what sorts of structures $\mathcal{A}$ and the space of states on it have. We shall see they actually admit algebraic structures. One could say the study of physics is the study of an algebra of observables and of the space of states defined upon it, so that is what we are going to do.

Algebra of Observables

Still following [1, 3], let us not see how and why $\mathcal{A}$ has an algebraic structure.

We can always add elements of $\mathcal{A}$. Given A and B, we define $A + B$ as the observable that yields

$$\omega(A + B) = \omega(A) + \omega(B), \tag{1.3.7}$$

for all states ω. To define λA for a real number $\lambda \in \mathbb{R}$ we can simply consider the function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(q) = \lambda q$ and define $\lambda A = f(A)$. Through this manner, one can show $\mathcal{A}$ is a real vector space.

Since we know how to compute functions of an observable, we can compute the square of an observable A^2. With this construction, $\mathcal{A}$ gains a natural product

$$A \circ B = \frac{1}{2}[(A + B)^2 - A^2 - B^2]. \tag{1.3.8}$$

This is known as the Jordan product. In practice, it is more convenient to assume the existence of an associative product $\cdot\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that

$$A \circ B = \frac{1}{2}[A \cdot B + B \cdot A]. \tag{1.3.9}$$

This associative product does not need to be commutative, although $\circ$ is. [1, 2, 50] give justification for why it is more interesting to work with $\cdot$ than with $\circ$. We shall take this associative product for granted.

We can still further enlarge the algebra $\mathcal{A}$ by allowing it to assume complex values. In this case, we should require that physical observables are “real”, in some sense. Hence, we assume $\mathcal{A}$ to possess a $*$ operation such that

$$(A^*)^* = A \quad \text{and} \quad (A \cdot B)^* = B^* \cdot A^*, \tag{1.3.10}$$

mimicking the Hermitian conjugate of matrices. Physical observables are then Hermitian observables, which satisfy $A^* = A$. Multiplication by a complex scalar can be defined in similarity with how we defined multiplication by a real scalar.

At last, we require the algebra to have the unit observable, $\mathbb{1}$. This is the observable $\mathbb{1} = f(A)$ for $f(q) = 1, \forall\, q \in \mathbb{R}$.

We thus find that it is reasonable to assume $\mathcal{A}$ has the following structure:

i. it is a complex vector space;

ii. it has a bilinear product $\cdot \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$;

iii. $\cdot$ is associative;

iv. there is an involution $* \colon \mathcal{A} \to \mathcal{A}$ such that $(A \cdot B)^* = B^* \cdot A^*$ and $(A^*)^* = A$.

This is the structure of a $*$-algebra. For simplicity, we overlooked the topological details of the construction—see [2, 3, 52] for those details that lead to a C^*-algebra—but it is possible to deal with the algebra of observables at this level, as done in [32]. We shall follow this simpler route.

One can show that all of these algebras can be realized as algebras of operators acting on some Hilbert space—see [7, 32]. However, commutative C^* algebras are equivalent to algebras of functions on some phase space (see [7]). Since classical theories live on a phase space, a theory is classical if, and only if, its algebra of observables is commutative. Meanwhile, a theory is a quantum theory if, and only if, its algebra of observables is not commutative.

Our goal is then to obtain a somehow natural, non-commutative algebra of observables for a Klein–Gordon field. This is the goal of the next chapter.

However, something is missing: a characterization of what is a state. Within our construction, a state becomes merely a normalized ($\omega(\mathbb{1}) = 1$), linear functional on $\mathcal{A}$. One can check this naturally arises from our previous characterization in experimental terms.

1.4 Reading Recommendations

Some common references on quantum field theory in curved spacetime are the books by Birrell and Davies [5], Fulling [18], Mukhanov and Winitzki [38], Parker and Toms [41], and Wald [61]. Most of them should discuss globally hyperbolic spacetimes, which are also discussed in the books by Hawking and Ellis [28] and Wald [59]. The algebraic approach is well motivated in the books by Alfsen and Shultz [2], Araki [3], and Strocchi [52].

Two

Algebraic Approach to Free Quantum Fields in Curved Spacetime

We begin by discussing some more causal structure properties in order to quantize the free Klein–Gordon field. This is done by means of the algebraic approach and we also explore some interesting states, such as vacua and thermal states. We conclude by describing the Unruh effect in Minkowski spacetime and different vacua in Schwarzschild spacetime.

2.1 Algebra of Observables

In order to proceed, it is useful to define a notion of lightcone for general spacetimes. This is encoded in the causal past and future of a set.

Causal Structure and Green Functions

Definition 2.1:

Let (M, g_{ab}) be a spacetime. Let $S \subseteq M$. We define the *causal future of* S as the set $J^+(S)$ defined by

$$J^+(S) = \{p \in M; \text{there is a future directed timelike curve } \lambda(t) \text{ with } \lambda(0) \in S \text{ and } \lambda(1) = p\}. \tag{2.1.1}$$

Similarly one defines the *causal past of* S, $J^-(S)$. ♠

Due to the good properties of a globally hyperbolic spacetime, it is possible to find interesting Green functions. These are functions (distributions, really) G such that

$$(\nabla_a \nabla^a - m^2) G(x, x') = \frac{1}{\sqrt{-g}} \delta^{(4)}(x, x'). \tag{2.1.2}$$

Notice that the function

$$Gj(x) = \int G(x, x') j(x') \sqrt{-g} \, \mathrm{d}^4 x' \tag{2.1.3}$$

is such that

$$(\nabla_a \nabla^a - m^2) Gj(x) = j(x), \tag{2.1.4}$$

where Gj has some boundary conditions that depend on the particular choice of G. This is also common in Electrodynamics, for example [58]. Hence, obtaining a Green function with some specific boundary conditions is sufficient to solve the Klein–Gordon equation under those boundary conditions with any source.

Two interesting Green functions are known as the advanced and the retarded Green functions. These are defined by the following boundary conditions:

- the advanced Green function G^- is defined by the property that* $\operatorname{supp} G^- j \subseteq J^-(\operatorname{supp} j)$;

*$\operatorname{supp} f$ denotes the support of f, *i.e.*, the set of all points x at which $f(x) \neq 0$.

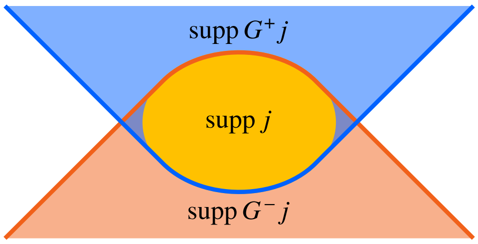

Figure 2.1: Causal diagrams illustrating the good causal behavior of solutions to the Klein–Gordon equation. The diagrams are drawn such that null geodesics are always at 45°. The supports of the advanced and retarded Green functions G^- and G^+ are shown.

- the retarded Green function G^+ is defined by the property that $\operatorname{supp} G^+ j \subseteq J^+(\operatorname{supp} j)$.

Hence, the advanced Green function propagates into the causal past of the source, while the retarded Green function propagates into the future of the source. This is pictured on Fig. 2.1.

In any globally hyperbolic spacetime, the existence of $G^\pm$ is ensured [21]. We can use them to define the commutator function

$$E = G^+ - G^-. \tag{2.1.5}$$

This function has the property that, on the spacetime manifold $M = \mathbb{R} \times \Sigma$ (something always possible to write in the presence of a Cauchy surface Σ), it holds that [13] (see also [23])

$$\partial_t E(t, \vec{\boldsymbol{x}}; t', \vec{\boldsymbol{x}}')\Big|_{t=t'} = -\delta^{(3)}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}') \quad \text{and} \quad E(t, \vec{\boldsymbol{x}}; t', \vec{\boldsymbol{x}}')\Big|_{t=t'} = 0, \tag{2.1.6}$$

the second equation being merely a statement that the commutator function vanishes on spacelike related events. We can then impose commutation relations by choosing

$$\left[\varphi(t, \vec{\boldsymbol{x}}), \varphi(t, \vec{\boldsymbol{x}}')\right] = 0, \tag{2.1.7}$$

$$\left[\partial_t\varphi(t, \vec{\boldsymbol{x}}), \partial_t\varphi(t, \vec{\boldsymbol{x}}')\right] = 0, \tag{2.1.8}$$

$$\left[\varphi(t, \vec{\boldsymbol{x}}), \partial_t\varphi(t, \vec{\boldsymbol{x}}')\right] = i\delta^{(3)}(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}')\mathbb{1}, \tag{2.1.9}$$

which are the usual equal-time commutation relations [14]. In a covariant notation, we can then write

$$[\varphi(x), \varphi(x')] = iE(x, x')\mathbb{1}. \tag{2.1.10}$$

One should notice that $E(x, x')$ vanishes if x and x' are spacelike related due to the support properties of the propagators, and hence Eq. 2.1.10 implements Einstein causality: operators on spacelike related regions commute.

In practice, E is actually a distribution. Indeed, in Minkowski spacetime translation invariance implies $E(x, x') = E(x - x')$ and one has [6]

$$E(x) = -\frac{1}{2\pi}\operatorname{sign}(x^0)\delta\left(-x^\mu x_\mu\right) + \frac{\operatorname{sign}(x^0)\Theta\left(-x^\mu x_\mu\right)J_1\left(m\sqrt{-x^\mu x_\mu}\right)}{4\pi\sqrt{-x^\mu x_\mu}}, \tag{2.1.11}$$

where sign is the sign function, Θ is the Heaviside step function, J_1 is the Bessel function and $x^\mu x_\mu$ stands for the invariant interval.

To avoid the mathematical difficulties of distributions, we shall do as usual with them and smear them against test functions $f \in C_0^\infty(M)$, which are the smooth functions of compact support. Hence, we write, for example,

$$\varphi(f) = \int \varphi(x) f(x) \sqrt{-g}\, \mathrm{d}^4 x\,. \tag{2.1.12}$$

We thus get

$$[\varphi(f_1), \varphi(f_2)] = iE(f_1, f_2)\mathbb{1}, \tag{2.1.13}$$

where

$$E(f_1, f_2) = \int E(x, x') f_1(x) f_2(x') \sqrt{-g(x)}\, \mathrm{d}^4x \sqrt{-g(x')}\, \mathrm{d}^4x' . \tag{2.1.14}$$

Algebra for the real Klein–Gordon field

It seems natural to build the algebra of observables out of the objects $\varphi(f)$ and their products. Hence, that is the path we are going to take. Since we have already imposed Eq. 2.1.13, we should start from there. This approach is similar to the one taken in [29].

The definition of $\varphi(f)$ as a smearing of φ with f leads us to impose

$$\varphi(f_1) + \lambda\varphi(f_2) = \varphi(f_1 + \lambda f_2), \tag{2.1.15}$$

for any $f_1, f_2 \in C_0^\infty(M)$ and any $\lambda \in \mathbb{C}$.

Furthermore, we want the quantum field to somehow solve the Klein–Gordon equation. Notice then that

$$\begin{aligned} 0 &= \int (\nabla_a\nabla^a - m^2)\varphi(x) f(x)\sqrt{-g}\, \mathrm{d}^4x\,, && (2.1.16a)\\ &= \int \varphi(x)(\nabla_a\nabla^a - m^2) f(x)\sqrt{-g}\, \mathrm{d}^4x\,, && (2.1.16b)\\ &= \varphi((\nabla_a\nabla^a - m^2)f), && (2.1.16c) \end{aligned}$$

which is thus another condition we impose on the algebra.

Finally, we also notice that

$$\begin{aligned} \varphi(f)^* &= \int \overline{\varphi}(x)\overline{f}(x)\sqrt{-g}\, \mathrm{d}^4x\,, && (2.1.17a)\\ &= \int \varphi(x)\overline{f}(x)\sqrt{-g}\, \mathrm{d}^4x\,, && (2.1.17b)\\ &= \varphi(\overline{f}), && (2.1.17c) \end{aligned}$$

which imposes that φ is a real field.

Notice that similar constructions could be performed for complex fields or higher spin fields. Nevertheless, our focus in these notes will remain in the real Klein–Gordon field.

In summary, the algebra of observables for the real Klein–Gordon field is generated by the objects $\varphi(f)$, $f \in C_0^\infty(M)$, and satisfies the conditions

i. $\varphi(f_1) + \lambda\varphi(f_2) = \varphi(f_1 + \lambda f_2)$, for all $f_1, f_2 \in C_0^\infty(M)$ and all $\lambda \in \mathbb{C}$;

ii. $\varphi((\nabla_a\nabla^a - m^2)f) = 0$ for all $f \in C_0^\infty(M)$;

iii. $\varphi(f)^* = \varphi(\overline{f})$ for all $f \in C_0^\infty(M)$;

iv. $[\varphi(f_1), \varphi(f_2)] = iE(f_1, f_2)\mathbb{1}$, for all $f_1, f_2 \in C_0^\infty(M)$.

2.2 Notable States

Vacua

It means little to have an algebra without at least some physically interesting states to go with it. We will begin by discussing the curved spacetime generalizations of the vacuum state of ordinary quantum field theory.

In ordinary quantum field theory, the vacuum is often thought of as the state of minimum energy, or the unique Poincaré-invariant state. Neither of these two classification are available in a general spacetime. Hence, we will need to find other manners of defining what is *a* (not *the*) vacuum state.

Firstly, let us notice that we can characterize a state by specifying how it acts on every observable. Within the algebra we just built we can characterize a state by providing its n-point correlation functions, or Wightman functions [51]. These are the functions given by

$$W_n(f_1, \dots, f_n) = \omega(\varphi(f_1) \cdots \varphi(f_n)). \tag{2.2.1}$$

The Minkowski vacuum correlation functions have an interesting property: they satisfy

$$W_{2n-1}(f_1, \dots, f_{2n-1}) = 0, \tag{2.2.2}$$

and

$$W_{2n}(f_1, \dots, f_{2n}) = \sum_{\text{pairings}} W_2(f_{i_1}, f_{i_2}) \cdots W_2(f_{i_{2n-1}}, f_{i_{2n}}). \tag{2.2.3}$$

This is known as Wick's theorem for the case of the Minkowski vacuum, but for us it will be the definition of a *Gaussian*, or *quasifree*, state. Our first requirement for a vacuum state is thus that it be Gaussian.

Gaussian states still have a lot of possibilities. To further restrict the notion of a vacuum, we will also impose that vacua should be pure states.

As we shall see on Chapter 3, this definition conforms with the traditional meaning of a vacuum being annihilated by all annihilation operators. This is meant in the sense that it will always be possible to represent the algebra of observables in a Fock space in which a vacuum is annihilated by all annihilation operators as defined in such Fock space.

Thermal States

While the vacuum is perhaps the most important state in quantum field theory (QFT) in flat spacetime, another important class of states are those in thermal equilibrium. In the algebraic approach, the notion of thermal equilibrium is given by means of the Kubo–Martin–Schwinger (KMS) condition [33, 35]. Our discussion is inspired by the one given by Raszeja [43, Sec. 2.3].

Suppose we have a system with finitely many degrees of freedom in contact with a thermal reservoir at inverse temperature $\beta = \frac{1}{T}$ (we take $k_B = 1$). Assuming the system to be in thermal equilibrium, its state is described by the density matrix

$$\rho = \frac{e^{-\beta H}}{Z}, \tag{2.2.4}$$

where $Z = \operatorname{tr}\left[e^{-\beta H}\right]$ is the partition function for the system and H its Hamiltonian.

We take a finite system to provide us with some motivation and intuition while avoiding the difficulties that occur in larger systems. In this simplified case we can work with density matrices, but in field theory we will not always have a privileged Hilbert space. Therefore, we would like to obtain a purely algebraic condition that expresses thermal equilibrium.

Thermal equilibrium is a notion closely related to the time evolution of a system. Hence, it is natural that to discuss equilibrium we need to consider some sort of time evolution.

In the context of finite quantum systems, this evolution is ruled by the Heisenberg equation of motion and, for a time-independent Hamiltonian, it is given by

$$A(t) = e^{itH} A(0) e^{-itH}, \tag{2.2.5}$$

where $A(t)$ is some observable at time t and we take $\hbar = 1$. Notice that we can also see this time evolution as a one-parameter group of automorphisms acting on the algebra $\mathscr{A}$. More specifically, we can write

$$\theta_t(A) = e^{itH} A e^{-itH} \tag{2.2.6}$$

to denote the time evolution of A by an amount t. While unusual in ordinary quantum mechanics, this notation will serve us well in the following.

Notice that the exponentials that occur on the definition of θ_t are similar to the exponential that occurs in the expression for a density matrix in thermal equilibrium, apart from the fact that ρ involves a real exponential and θ_t involves imaginary exponentials. It is tempting, however, to consider an analytic continuation of θ_t to the complex t plane. If this is possible, then notice that, given $A, B \in \mathcal{A}$, we have

$$\omega_\rho(BA) = \operatorname{tr}[BA\rho], \tag{2.2.7a}$$

$$= \frac{1}{Z} \operatorname{tr}\left[BAe^{-\beta H}\right], \tag{2.2.7b}$$

$$= \frac{1}{Z} \operatorname{tr}\left[Ae^{-\beta H}B\right], \tag{2.2.7c}$$

$$= \frac{1}{Z} \operatorname{tr}\left[Ae^{-\beta H}Be^{+\beta H}e^{-\beta H}\right], \tag{2.2.7d}$$

$$= \frac{1}{Z} \operatorname{tr}\left[A\theta_{i\beta}(B)e^{-\beta H}\right], \tag{2.2.7e}$$

$$= \operatorname{tr}\left[A\theta_{i\beta}(B)\rho\right], \tag{2.2.7f}$$

$$= \omega_\rho(A\theta_{i\beta}(B)). \tag{2.2.7g}$$

In the previous expressions, ω_ρ is the state defined through $\omega_\rho(A) = \operatorname{tr}[A\rho]$.

Eq. (2.2.7) leads one to the general expression

$$\omega(BA) = \omega(A\theta_{i\beta}(B)), \tag{2.2.8}$$

where θ_t can be any one-parameter group of automorphisms in the algebra, as long as it admits a suitable analytic extension. This property, known as the KMS condition, was originally used in [22] as a definition for equilibrium states in an algebraic setting.

It should be pointed out that while Eq. (2.2.8) is sufficient for C^*-algebras, general $*$-algebras are more subtle and require other additional conditions for the expectation values of the form $\omega(A_1 \cdots A_n)$ with $n > 2$ [29]. Nevertheless, our main focus is on Gaussian states, which are completely determined by the two-point function. Hence, for our purposes, Eq. (2.2.8) is enough. We say that a state satisfying Eq. (2.2.8) is a KMS state for the one-parameter group of automorphisms θ_t at inverse temperature β.

There is an important difference between how we defined vacua and how we defined KMS states. Notice that a vacuum for us is a state that is Gaussian and pure. These properties are related exclusively to the state and the algebra of observables, and are independent of any other choice or input from the physicist. Hence, given a state on an algebra of observables, one can immediately say whether it is a vacuum. KMS states, on the other hand, depend on the choice of a group of automorphisms. In other words, it depends on a choice of time evolution. Two different choices of time evolution might disagree on whether a given state is a KMS state. Hence, thermal equilibrium depends on something in addition to the state itself. In quantum field theory in curved spacetime (QFTCS) this is relevant because different observers will have different definitions of time evolution. Hence, a state might be a KMS state for an observer, but not for another. Furthermore, two different observers might agree that a state is a KMS state, but disagree on what is its temperature. This is well illustrated by the Unruh effect, which shows that different observers can perceive the same vacuum state as having different temperatures depending on their acceleration.

Hadamard States

While we have so far focuses on linear fields, computing expectation values of nonlinear field combinations is essential in physics. For example, the stress-energy-momentum tensor is nonlinear in the quantum fields.

The main difficulty with nonlinear objects is the fact that the fields are distributions, and distributions not always admit nonlinear operations to be performed upon them. For example, while $\delta(x)$ makes sense, $\delta(x)^2 = \delta(x)\delta(0)$ does not.

Notice, however, that this is only a problem in the coincidence limit

$$\lim_{y\to x} \delta(x)\delta(y). \tag{2.2.9}$$

Similarly, the issue with the n-point functions will only arise in the limit

$$\lim_{y\to x} \omega(\varphi(x)\varphi(y)), \tag{2.2.10}$$

which thus refers to the ultraviolet (UV) behavior of the state ω.

In Minkowski spacetime, this is dealt with by requiring operator expressions to be normal ordered, which means essentially ignoring unphysical infinities when they are obviously not there. Alternatively, one could say that the expectation value that really matters is not the expectation value itself, but rather the vacuum-subtracted expectation value, which has thus a good behavior in the UV limit.

In curved spacetime, we shall thus require of physical states that, in the UV limit, they behave in a manner similar to the Minkowski vacuum. Thanks to the equivalence principle, this expression can be given a meaningful sense. In fact, it admits a beautiful formulation within microlocal analysis, as reviewed in [1, 9, 53].

It should be pointed out that, in a stationary spacetime—*i.e.*, a spacetime with a notion of timelike symmetry—there is at most one stationary Hadamard vacuum [31]. Furthermore, one should notice that the notion of state is global, and hence might depend on features spacelike related to the point under consideration. For example, Schwarzschild spacetime admits at least three different physical vacua depending on which regions of Schwarzschild spacetime are being considered.

2.3 The Unruh Effect

Once the basis of the theory has been outlined, let us consider the case of acceleration-induced thermality in Minkowski spacetime. We shall do this discussion in four different ways throughout the text, since each of them can play different roles in explaining the physics going on. Furthermore, since experimental probing of the Unruh effect has only recently begun (the first claim of a direct observation has been made in [34]), the existence of many paths to the same conclusion helps us to understand why it must be true.

Before we get to the actual calculations, let us explain what we are about to do. The Fulling–Davies–Unruh effect [11, 19, 55], often called simply the Unruh effect, is the result that "for a [. . .] quantum field in its vacuum state in Minkowski spacetime, an observer with uniform acceleration a will feel that he is bathed by a thermal distribution of quanta of the field at temperature T given by $k_B T = \frac{\hbar a}{2\pi c}$" [56]. It consists of a prediction made with QFTCS methods in flat spacetime, and challenges one's usual understanding of the meaning of "particles", since different observers are shown to have different particle interpretations of the same physical state. More details will be given in the following discussion, and even more can be found in the review [10].

In all of the following approaches, we consider Minkowski spacetime, $\mathcal{M} = (\mathbb{R}^4, \eta_{ab})$. The line element is given in Cartesian coordinates by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 . \tag{2.3.1}$$

When analyzing the Unruh effect, our interest will not be in the entire Minkowski spacetime, but rather on a region known as the right Rindler wedge [44]. We will denote it by

$$R = \left\{(t, x, y, z) \in \mathbb{R}^4; x > |t|\right\}. \tag{2.3.2}$$

This region can be understood as a globally hyperbolic spacetime in its own right, and it is particularly useful to mimic some properties found in black hole spacetimes. Many properties of the Rindler spacetime are reviewed, for example, in the books [15, 45]. Since the literature on this spacetime is vast, we shall state some of its properties without proof.

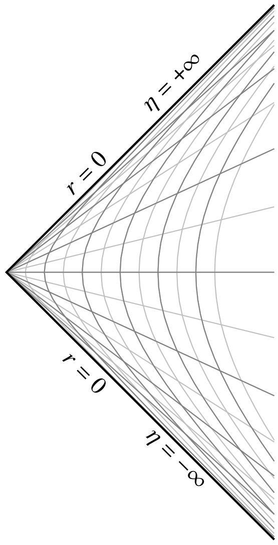

Figure 2.2: Depiction of how the Rindler coordinates given on Eq. (2.3.3) cover the right Rindler wedge. The hyperbolae are curves of constant r, while the straight lines are curves of constant η.

It is convenient for our purposes to cover the Rindler spacetime using the so-called Rindler coordinates [44, 45]. We define them through

$$t = r \sinh a\eta \quad \text{and} \quad x = r \cosh a\eta, \tag{2.3.3}$$

for constant $a > 0$. They are illustrated on Fig. 2.2. This definition leads to the line element

$$\mathrm{d}s^2 = -a^2 r^2 \,\mathrm{d}\eta^2 + \mathrm{d}r^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,. \tag{2.3.4}$$

Radar coordinates [36], which use $r = a^{-1}e^{a\xi}$, are also common in the literature.

Notice that surfaces of constant η are Cauchy surfaces. Furthermore, the spacetime is static with Killing field $\left(\frac{\partial}{\partial\eta}\right)^a$. Since a four-dimensional spacetime has at most ten Killing fields and this is a section of Minkowski spacetime—which has its Killing fields as Poincaré transformations—we know that this Killing field is somehow related to Poincaré transformations. It turns out it is simply the generator of boosts along the x direction. It also happens to be proportional to the four-velocities of observers with constant proper acceleration. In fact, the parameter a introduced earlier is the proper acceleration of the observers moving along the locus with $\eta_{ab}\left(\frac{\partial}{\partial\eta}\right)^a\left(\frac{\partial}{\partial\eta}\right)^b = -1$.

The orbits induced by Lorentz boosts on Minkowski spacetime are illustrated on Fig. 2.3 on the next page.

Our first derivation follows the algebraic spirit we have been establishing so far. We follow the discussion given by [29].

To obtain the desired QFT, we can simply consider the algebra of observables $\mathcal{A}(\mathcal{M})$, but now restrict it to only (linear combinations, products, and involutions of) observables of the form $\varphi(f)$ with $\operatorname{supp} f \subseteq R$ (a condition that implies $\operatorname{supp} f \cap \partial R = \varnothing$). Through this restriction, we get to the subalgebra $\mathcal{A}(R) \subseteq \mathcal{A}(\mathcal{M})$.

If ω is a state on $\mathcal{A}(\mathcal{M})$, it is also a state on $\mathcal{A}(R)$—after all, we are simply considering less observables. Hence, the Minkowski vacuum defines a state on $\mathcal{A}(R)$. We desire to characterize it.

For simplicity, let us assume a massless field. In this case, we know that the two-point function is given by

$$W_2(x_1, x_2) = \operatorname*{w-lim}_{\varepsilon\to 0^+} \frac{1}{4\pi^2(x_1^\mu - x_2^\mu - i\varepsilon T^\mu)(x_\mu^1 - x_\mu^2 - i\varepsilon T_\mu)}, \tag{2.3.5}$$

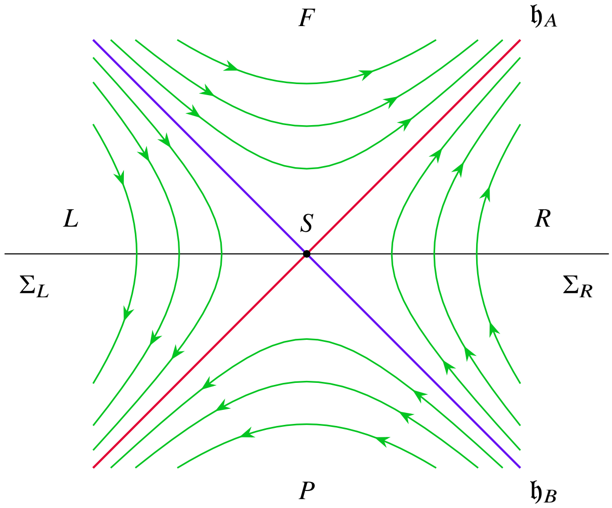

Figure 2.3: Orbits induced by Lorentz boosts on 1 + 1-dimensional Minkowski spacetime. Notice that on the left and right Rindler wedges (L and R, respectively) the orbits are timelike, while they are spacelike on the remaining wedges. On the null hypersurfaces $\mathfrak{h}_A = \left\{(t, x, y, z) \in \mathbb{R}^4; t = x\right\}$ and $\mathfrak{h}_B = \left\{(t, x, y, z) \in \mathbb{R}^4; t = -x\right\}$ that separate the wedges the orbits are also null. The spacelike submanifold $S = \mathfrak{h}_A \cap \mathfrak{h}_B$ is comprised of fixed points of the isometry orbits. Σ_R (resp. Σ_L) is a Cauchy surface for the right (left) Rindler wedge.

which can be derived using the expressions given in App. V.2 of [6]. w-lim is the weak limit and T^a stands for any future-directed timelike vector. Taking it to be $\left(\frac{\partial}{\partial t}\right)^a$, we find that, in Cartesian coordinates,

$$W_2(x_1, x_2) = \operatorname*{w-lim}_{\varepsilon \to 0^+} \frac{1}{4\pi^2\left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - (t_1 - t_2 - i\varepsilon)^2\right]}. \tag{2.3.6}$$

These expressions hold for the entire Minkowski spacetime, not only on the right Rindler wedge. In particular, notice they mean there are correlations between the left and right Rindler wedges, where the left wedge is defined as $L = \left\{(t, x, y, z) \in \mathbb{R}^4; x < -|t|\right\}$. When restricting the Minkowski vacuum to the right Rindler wedge, we drop these correlations, meaning the state can no longer be pure.

It turns out the state is not only mixed, but also a KMS state at inverse temperature $\beta = \frac{2\pi}{a}$ for the isometry $\eta \mapsto \eta + \tau$, where τ is some arbitrary parameter. In other words, it is thermal with respect to the time-evolution prescribed by accelerated observers, with temperature proportional to the acceleration.

To see this in the algebraic approach, we must show that the state satisfies the KMS condition. The Minkowski vacuum is Gaussian, and hence we essentially want to show that [29]

$$\omega(\varphi(f)\theta_{\tau+i\beta}(\varphi(g))) = \omega(\theta_\tau(\varphi(f))\varphi(g)), \tag{2.3.7}$$

where θ denotes the isometry $\eta \mapsto \eta + \tau$. To do so, one can use Eqs. (2.3.3) and (2.3.6) on the preceding page and on the current page to show that

$$W_2(x_1, \theta_{\tau+i\beta}(x_2)) = W_2(\theta_\tau(x_2), x_1). \tag{2.3.8}$$

This can be shown by direct calculation.

2.4 The Hawking Effect

An effect similar to the Unruh effect, but that predates it historically, is the Hawking effect [26, 27]. It concerns the observation of particle creation in a spacetime containing a black hole. We shall discuss it without much detail (more information and detailed calculations can be found in references such as [27, 59–61]).

This time, we shall consider three different vacua and discuss qualitatively the results obtained for each one of them.

The Hartle–Hawking Vacuum

Our first scenario is as following. For simplicity, we consider an eternal Schwarzschild black hole. Its Penrose diagram is depicted in Fig. 2.4. In this case, there is exactly one stationary Hadamard vacuum, called the Hartle–Hawking state [25]. Detailed properties about it and other vacua that we shall mention can be found in the book [17].

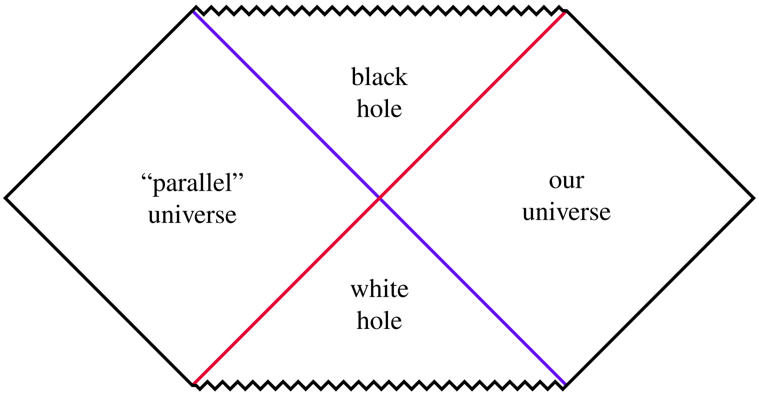

Figure 2.4: Penrose diagram for an eternal black hole. The zigzagged lines depict physical singularities. Each point inside the diagram is actually a 2-sphere.

The Hartle–Hawking vacuum is a thermal equilibrium state at the Hawking temperature,

$$T_H = \frac{\hbar c^3}{8\pi G M k_B}. \tag{2.4.1}$$

This state can be thought to model a "black hole in a box", for it involves not only thermal radiation coming from the black hole region, but incoming modes from infinity. Due to the fact that this effect is derived mimicking the Unruh effect—as done in [61]—this is an instance of the Unruh effect in curved spacetimes.

The Unruh Vacuum

A second interesting vacuum in the Schwarzchild spacetime is the Unruh vacuum, which physically corresponds to the situation in which the black hole arises from gravitational collapse of a star, for example. This is pictured in Fig. 2.5 on the next page.

This physical situation can also be pictured as an eternal Schwarzschild black hole, but with different boundary conditions. As depicted in Fig. 2.6 on the following page, we do not need to consider the white hole's event horizon when considering a black hole formed by gravitational collapse. Therefore, this time, when we look for a stationary Hadamard state, the horizon associated with the white hole should be ignored.

The vacuum now cannot be the Hartle–Hawking vacuum, for it involves modes coming from the white hole, which simply does not exist in our present scenario. Instead, we get the Unruh vacuum. This state fails to be Hadamard at the white hole's event horizon, but that is not an issue, for the white hole does not exist in this physical situation anyway.

The Unruh vacuum leads to a different prediction when compared to the Hartle–Hawking vacuum. It is still a thermal state at the same Hawking temperature of Eq. (2.4.1), but this time only modes coming from the direction of the black hole are thermal. There is no contribution due to incoming modes from infinity. This is known as the Hawking effect, and was originally predicted in [26, 27].

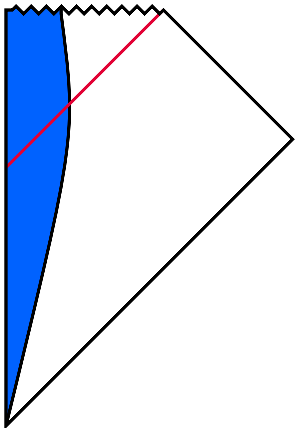

Figure 2.5: Penrose diagram for a black hole arising from gravitational collapse. The zigzagged line depict a physical singularity. Each point inside the diagram is actually a 2-sphere.

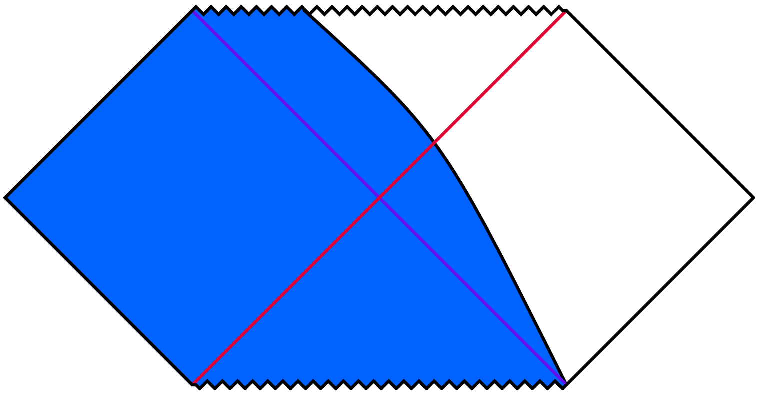

Figure 2.6: Penrose diagram for a black hole arising from gravitational collapse, as seen a piece of the eternal Schwarzschild black hole. Notice the blue matter covers the purple horizon that bounded the "white hole" and the "parallel universe" in Fig. 2.4 on the preceding page. The zigzagged lines depict physical singularities. Each point inside the diagram is actually a 2-sphere.

The Boulware Vacuum

At last, one may question whether there is thermal radiation coming from planets or stars due to quantum effects. The answer is no, and the reason is that the different physical situation given by such objects leads to a different physical vacuum.

The Penrose diagrams for this new physical scenario are given in Figs. 2.7 and 2.8 on the facing page. Notice that the absence of horizons implies the physical vacuum should now not include any modes trespassing the horizons. Thus, the Hartle–Hawking and Unruh vacua are already excluded.

The stationary Hadamard vacuum we can get now is the so-called Boulware vacuum. It is a vacuum at zero temperature, in the sense that a static observer sees no particles are all coming from either the object or from infinity. This state is not admissible in the previous situations because it fails to be Hadamard at the future and past event horizons, which is meaningless when the physical scenario includes no horizons.

As a consequence, no Hawking or Unruh radiation is observed near a static planet or star, even though an observer standing on top of the planet is technically accelerated. Hence, notice how the equivalence principle is not enough to derive thermal effects in curved spacetimes, for the vacua themselves are global constructions.

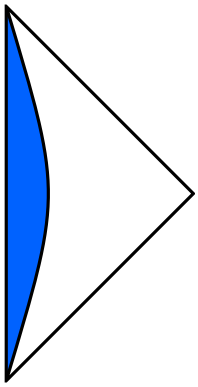

Figure 2.7: Penrose diagram for a static object, such as a star or a planet. Each point inside the diagram is actually a 2-sphere.

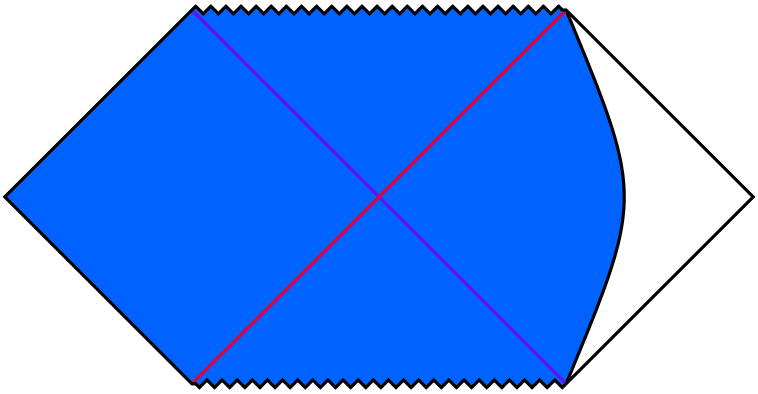

Figure 2.8: Penrose diagram for a static object, such as a star or a planet, as seen a piece of the eternal Schwarzschild black hole. Notice the blue matter covers the both the horizons depicted in Fig. 2.4 on page 15. The zigzagged lines depict physical singularities. Each point inside the diagram is actually a 2-sphere.

2.5 Criticism of the Algebraic Approach

While the algebraic approach is powerful and general, it should be recognized that it is difficult to perform calculations with it. While the definition of a KMS state is sufficiently clear and the Minkowski vacuum is sufficiently simple so that the computation of the Unruh effect is not difficult, most calculations within the algebraic approach might not be straightforward.

Its advantage, nevertheless, lies on the myriad of situations it can be applied to. It provides a satisfactory, and mathematically rigorous, definition of a QFT in an arbitrary globally hyperbolic spacetime. It also has no preference of notion of "particles" nor does it require special symmetries or congruences of observers. Therefore, it allows one to prove theorems and study QFTCS in a fairly general setting.

Hence, it should be seen as a technical tool intended to explore what is QFT, rather than a calculation method to obtain quantitative results.

2.6 Reading Recommendations

Most of our discussion in this chapter is inspired by the review by Hollands and Wald [29]. Our definition of Gaussian states follows Khavkine and Moretti [32]. A good reference on understanding the KMS condition is the PhD dissertation by Raszeja [43]. Hadamard states are discussed in further detail in many references, such as the review by Khavkine and Moretti [32], and there is a pedagogical summary in the author's master thesis [1]. The books by Birrell and Davies [5] and Frolov and Novikov [17] discuss the three different vacua in Schwarzschild spacetime.

Three

Fock Space Representations

We discuss how to connect the algebraic approach with the more usual Fock space approach to quantum field theory. Using it, we rederive the Unruh effect by means of a Bogolyubov transformation.

3.1 Algebraic Representations

Usually, quantum mechanics is discussed in terms of Hilbert spaces, not in terms of algebras. As a consequence, the procedure we have taken so far may seem a bit awkward. We shall now discuss how to recover the Hilbert space approach.

When working with a Hilbert space $\mathcal{H}$, the observables are linear operators acting on $\mathcal{H}$. We denote this space of linear operators as $\mathcal{L}(\mathcal{H})$. We would then like to relate the algebra of observables, $\mathscr{A}$, to $\mathcal{L}(\mathcal{H})$ somehow. More specifically, we would like to have a linear map $\rho\colon \mathscr{A} \to \mathcal{L}(\mathcal{H})$ such that $\rho(AB) = \rho(A)\rho(B)$ and $\rho(A^*) = \rho(A)^*$. ρ is said to be a representation of the algebra $\mathscr{A}$ on the Hilbert space $\mathcal{H}$. Hence, a representation of an algebra on a Hilbert space is a "copy" of the algebra in the operators acting on the Hilbert space. This copy may or may not be faithful: nothing prevents a representation from assigning the same operator on $\mathcal{L}(\mathcal{H})$ to different elements of $\mathscr{A}$. If the representation is one-to-one, it is said to be faithful.

Suppose now we are given an algebra $\mathscr{A}$ and a state $\omega\colon \mathscr{A} \to \mathbb{C}$. It is possible to perform a procedure known as the Gelfand–Naimark–Segal (GNS) construction that yields a Hilbert space $\mathcal{H}$ on which one can represent the algebra $\mathscr{A}$ by means of operators acting on $\mathcal{H}$. In other words, the GNS construction takes an algebra $\mathscr{A}$ and some state ω and yields all of the elements of a representation $\rho\colon \mathscr{A} \to \mathcal{L}(\mathcal{H})$.

The technical details of the GNS construction theorem vary depending on the assumptions taken. [32]'s Theorem 5.1.13 is the theorem for unital $*$-algebras (*i.e.*, $*$-algebras which have a unit element), while most books on C^*-algebras discuss the C^* case—which is the one usually meant when one speaks of the GNS construction. We will omit the details and focus on the physical meaning of the theorem.

Specifically, given a $*$-algebra satisfying some assumptions and some chosen state ω on said algebra, the GNS construction yields us

i. a Hilbert space $\mathcal{H}$;

ii. a representation $\rho\colon \mathscr{A} \to \mathcal{L}(\mathcal{H})$;

iii. a vector $|\omega\rangle \in \mathcal{H}$.

This triple has the property that $\omega(A) = \langle\omega|\rho(A)\omega\rangle$ for all $A \in \mathscr{A}$, and hence $|\omega\rangle$ represents ω as a vector in the Hilbert space. It also holds that*

$$\{\rho(A)\,|\omega\rangle\,;A \in \mathscr{A}\} = \mathcal{H}. \tag{3.1.1}$$

The GNS triple is also unique up to unitary isomorphism, *i.e.*, other triples with the same properties are related to the GNS triple by a unitary transformation. Do notice, however, that the GNS triple assumes a state underlying its construction, and different choices of states may lead to unequivalent representations.

*More precisely, the left-hand side of Eq. (3.1.1) is dense on the Hilbert space.

One may wonder whether two representations of the same algebra $\mathcal{A}$ are always ensured to be equivalent. In other words, suppose $(\mathcal{H}_1, \rho_1)$ and $(\mathcal{H}_2, \rho_2)$, $\rho_i : \mathcal{A} \to \mathcal{L}(\mathcal{H}_i)$, are representations of $\mathcal{A}$. Is it always possible to find a unitary transformation $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\rho(A)_2 = U\rho_1(A)U^*$? This is the most basic requirement for the two Hilbert spaces to yield the same theory. If we can't find such a transformation, then the inner products in a Hilbert space might have values different from those on the other Hilbert space. Hence, they can lead to different descriptions.

For quantum systems with a finite number of degrees of freedom, this question is answered by the Stone–von Neumann theorem [see, for example, 24, Theorem 14.8], which ensures that all representations of the canonical commutation relations (CCR) for a system with finitely many degrees of freedom are equivalent*. However, the theorem fails in the case of infinitely many degrees of freedom, *i.e.*, for field theory. Therefore, we might get non-equivalent Hilbert spaces.

The issue with getting non-equivalent Hilbert spaces is we cannot tell which one is the "correct" description. In some situations, symmetry considerations might allow us to pick a preferred Hilbert space out of all possible ones. For example, in Minkowski spacetime there is a single Poincaré-invariant state, the Minkowski vacuum. Hence, it seems natural to pick the Hilbert space obtained by using the GNS construction with the Minkowski vacuum. This leads one to the usual treatment of quantum field theory (QFT) given, *e.g.*, in [62]. Similar comments are applicable to QFT in stationary spacetimes such as Schwarzschild or De Sitter spacetimes, but not in a general curved spacetime.

Even in the occasion two different representations are equivalent, we should remark they can still have different interpretations. For example, in QFT, the Hilbert space is often taken to be a Fock space, which has a natural interpretation in terms of "particles". The notions of "particle" associated to the two equivalent Fock spaces might not be the same, and hence the unitary transformation will not preserve particle number. This occurs, for example, in Friedmann–Lemaître–Robertson–Walker universes with compact Cauchy surfaces. Nevertheless, there is no issue: we are interested in a quantum theory of fields, not of particles. However, it makes it clear that careless dependence on a Fock space might mix the actual physical content of the theory with misconceptions due to a belief in "particles" as fundamental entities.

3.2 Fock Representations in Stationary Spacetimes

Many spacetimes of interest end up being stationary or asymptotically stationary. For example, the Kerr–Newman family, De Sitter spacetime, Minkowski spacetime and many more are stationary. When working in these spacetimes, one can exploit the available stationary symmetry when doing quantum field theory in curved spacetime (QFTCS). In this appendix, we will describe a qualitative and "handwaving" understanding of the role of symmetry. More detailed expositions can be found in the discussions by Khavkine and Moretti [32, Sec. 5.2.7], Panangaden [40], and Wald [61, Sec. 4.3] and in the original papers by Ashtekar and Magnon [4] and Kay [30].

Let us begin by considering quantum fields in Minkowski spacetime. In Minkowski spacetime, it is common to discuss about creating and annihilating particles on a given state by using ladder operators. These operators are defined by means of the Fourier decomposition of the Klein–Gordon field. Namely, one can write

$$\varphi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(a_{\vec{p}} e^{ip\cdot x} + a^\dagger_{\vec{p}} e^{-ip\cdot x} \right) \frac{\mathrm{d}^3 p}{\sqrt{2\omega_{\vec{p}}}}, \tag{3.2.1}$$

where

$$\omega_{\vec{p}} = +\sqrt{\|\vec{p}\|^2 + m^2} \tag{3.2.2}$$

and we choose conventions such that

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = \delta^{(3)}(\vec{p} - \vec{q}). \tag{3.2.3}$$

*Technically, the Stone–von Neumann theorem is a statement about the Weyl algebra, which corresponds to an exponentiated version of the CCR algebra [see 61, Chap. 2, for details]. While the CCR algebra is merely a $*$-algebra, the Weyl algebra is a C^*-algebra, and hence it "behaves better" from a mathematical perspective.

Notice that Eq. (3.2.1) on the facing page decomposes the field φ in terms of positive and negative frequencies. In other words, in terms of solutions associated with positive and negative energies. This difference between positive and negative energies is then used to define the creation and annihilation operators.

It should be remarked that ortochronous Poincaré transformations never flip the sign of the time-component of a four-vector. Hence, all inertial observers in Minkowski spacetime always agree on the sign of the energy of a given particle. As a consequence, all inertial observers agree on the decomposition given on Eq. (3.2.1) on the preceding page. It follows that they all agree that all annihilation operators annihilate the Minkowski vacuum and they all agree on how many particles there are in a given state of the quantum field.

From this discussion, we can already conclude that, in some sense, particles are an "energy-dependent concept". The separation between positive and negative energy is literally the way we usually define the ladder operators used in QFT in flat spacetime to create and annihilate particles. Loosely speaking, if two observers have different notions of what is energy, they might have two different notions of what is a particle.

The natural question to ask is then: what is energy? Intuitively, we can understand energy as being the Noether charge associated with time-translation symmetry—*i.e.*, energy is the conserved quantity induced by a timelike Killing field.

We now know what to expect. In a stationary spacetime, we have a timelike Killing field. Hence, in some sense we have an available notion of energy. This notion of energy can then be used to induce a preferred notion of particles, which leads us to a natural choice of Fock space. Formally, one defines the vacuum $|0\rangle$ by imposing it is annihilated by all annihilation operators and defines every other state in the Fock space by applying creation operators. See the previously mentioned references for a more rigorous approach.

There is, however, an interesting issue with this discussion. Consider Minkowski spacetime once again. The Minkowski vacuum is the unique Poincaré invariant state. Nevertheless, if we chose to restrict our attention to the right Rindler wedge as necessary when discussing the Unruh effect, then we would also be dealing with a different stationary spacetime and would be able to construct a vacuum that is invariant under the boost symmetry. This is known as the Rindler vacuum. Do these two states coincide?

They do not. By construction, the notion of time employed in the definition of the Rindler vacuum is the notion of proper time of an accelerated observer. On the other hand, Minkowski vacuum is built upon the notion of time as defined by inertial observers. Hence, the Rindler vacuum corresponds to the quantum state of the field in which an accelerated observer would see no particles. The Minkowski vacuum corresponds to the state of the field in which an inertial observer would see no particles. The Unruh effect proves that these two states do not coincide.

There are still more striking differences. The boost symmetries of Minkowski spacetime have a geometric structure known as a "bifurcate Killing horizon". Roughly speaking, this means the Killing field becomes null on a pair of crossing hypersurfaces, as depicted on Fig. 2.3 on page 14. More detailed definitions can be found, *e.g.*, in the discussions by Kay and Wald [31, Sec. 2] and Wald [61, Sec. 5.2]. Other examples of spacetimes with Killing horizons are Schwarzschild, De Sitter, Schwarzschild–De Sitter, and Kerr spacetimes, among others. It was shown by Kay and Wald [31] that spacetimes with such a structure admit at most one quasifree state that is both stationary and Hadamard.

Since both the Minkowski and Rindler vacua are stationary quasifree states, one of them must fail to be Hadamard. It does happen that the Rindler vacuum has an unphysical build up of energy near the null hypersurfaces $t = \pm x$ [see 5, Eq. (6.157)].

Notice that this uniqueness result does not imply existence, and our earlier statement about existence of stationary states does not imply they are Hadamard. In fact, Kay and Wald [31] have also shown there are no stationary Hadamard states on Kerr or Schwarzschild–De Sitter spacetimes.

3.3 Bogolyubov Transformations and the Unruh Effect

Next we shall rederive the Unruh effect by using Fock space techniques. To do so, we will for simplicity work in $1+1$ dimensions with a massless field, which makes the computations easier because the theory becomes conformal. We follow the discussion by Mukhanov and Winitzki [38].

The action is

$$S=\frac{1}{2}\int g^{ab}\nabla_a\varphi\nabla_b\varphi\sqrt{-g}\,\mathrm{d}^2x\,. \tag{3.3.1}$$

In inertial coordinates, we can write the action as

$$S=\frac{1}{2}\int -\left(\frac{\partial\varphi}{\partial t}\right)^2+\left(\frac{\partial\varphi}{\partial x}\right)^2\mathrm{d}t\,\mathrm{d}x\,. \tag{3.3.2}$$

In radar coordinates [36], defined as

$$t=a^{-1}e^{a\xi}\sinh a\eta \quad\text{and}\quad x=a^{-1}e^{a\xi}\cosh a\eta, \tag{3.3.3}$$

the line element takes the form

$$\mathrm{d}s^2=-e^{2a\xi}\,\mathrm{d}\eta^2+e^{2a\xi}\,\mathrm{d}\xi^2\,, \tag{3.3.4}$$

and hence the action reads

$$S=\frac{1}{2}\int -\left(\frac{\partial\varphi}{\partial \eta}\right)^2+\left(\frac{\partial\varphi}{\partial \xi}\right)^2\mathrm{d}\eta\,\mathrm{d}\xi\,. \tag{3.3.5}$$

The equations of motion are then

$$\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}=0 \tag{3.3.6}$$

and

$$\frac{\partial^2\varphi}{\partial \eta^2}-\frac{\partial^2\varphi}{\partial \xi^2}=0. \tag{3.3.7}$$

If we define $u=t-x$, $v=t+x$, $U=\eta-\xi$, and $V=\eta+\xi$, we find that the solutions to the equations of motion must be of the forms

$$\varphi(t,x)=f_R(u)+f_L(v) \tag{3.3.8}$$

and

$$\varphi(\eta,\xi)=g_R(U)+g_L(V). \tag{3.3.9}$$

Notice that

$$\mathrm{d}s^2=-\,\mathrm{d}u\,\mathrm{d}v=-e^{2a(V-U)}\,\mathrm{d}U\,\mathrm{d}V\,. \tag{3.3.10}$$

In order to avoid having terms such as $\mathrm{d}u^2$, it must be true that u and v are each functions of only U or V each. It turns out that [see, *e.g.*, 61, p. 110] $u=u(U)$ and $v=v(V)$. More specifically, we have [38, Eq. (8.25)]

$$u(U)=-\frac{e^{-aU}}{a} \quad\text{and}\quad v(V)=+\frac{e^{-aV}}{a}. \tag{3.3.11}$$

For more on this system of coordinates, see [36, 38].

This discussion lets us see that Eqs. (3.3.8) and (3.3.9) allows us to decompose the field in left-moving and right-moving components. We shall then focus only on the right-moving components and know the left-moving components work analogously.

We can then decompose the field in Fourier modes according to

$$\varphi(t,x)=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\left[a_\omega e^{-i\omega(t-x)}+a_\omega^\dagger e^{+i\omega(t-x)}\right]\frac{\mathrm{d}\omega}{\sqrt{2\omega}}+\text{left-moving} \tag{3.3.12}$$

or

$$\varphi(\eta,\xi)=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\left[b_\Omega e^{-i\Omega(\eta-\xi)}+b_\Omega^\dagger e^{+i\Omega(\eta-\xi)}\right]\frac{\mathrm{d}\Omega}{\sqrt{2\Omega}}+\text{left-moving}. \tag{3.3.13}$$

It can be verified that

$$\left[a_\omega,a_{\omega'}^\dagger\right]=\delta(\omega-\omega') \quad\text{and}\quad \left[b_\Omega,b_{\Omega'}^\dagger\right]=\delta(\Omega-\Omega'). \tag{3.3.14}$$

Our questions is then how to relate these two descriptions. We know that an inertial observer would measure the number of particles of frequency ω in the vacuum as being

$$\left\langle a_{\omega}^{\dagger} a_{\omega} \right\rangle = 0, \tag{3.3.15}$$

for a_ω annihilates the Minkowski vacuum $|0\rangle_M$. However, what happens from the point of view of an accelerated observer, who has as creation and annihilation operators $b_\Omega^\dagger$ and b_Ω rather than $a_\omega^\dagger$ and a_ω?

In the Rindler wedge, the ladder operators must somehow be related to the inertial ladder operators, *i.e.*,

$$b_\Omega = \int_0^{+\infty} \alpha_{\Omega\omega} a_\omega - \beta_{\Omega\omega} a_\omega^\dagger \, \mathrm{d}\omega \, . \tag{3.3.16}$$

The normalization condition $[b_\Omega, b_{\Omega'}^\dagger] = \delta(\Omega - \Omega')$ implies

$$\int_0^{+\infty} \alpha_{\Omega\omega} \alpha_{\Omega'\omega}^* - \beta_{\Omega\omega} \beta_{\Omega'\omega}^* \, \mathrm{d}\omega = \delta(\Omega - \Omega'). \tag{3.3.17}$$

Eq. (3.3.16) is known as a Bogolyubov transformation. Using it on Eq. (3.3.13) on the preceding page and comparing with Eq. (3.3.12) on the facing page, we find that

$$\frac{e^{-i\omega u}}{\sqrt{\omega}} = \int_0^{+\infty} \left[\alpha_{\Omega\omega} e^{-i\Omega U} - \beta_{\Omega\omega}^* e^{+i\Omega U}\right] \frac{\mathrm{d}\Omega}{\sqrt{\Omega}}. \tag{3.3.18}$$

Notice then that

$$\int_{-\infty}^{+\infty} \frac{e^{-i\omega u \pm i\Omega' U}}{\sqrt{\omega}} \mathrm{d}U = \int_{-\infty}^{\infty} \int_0^{+\infty} \left[\alpha_{\Omega\omega} e^{-i(\Omega \mp \Omega')U} - \beta_{\Omega\omega}^* e^{+i(\Omega \pm \Omega')U}\right] \frac{\mathrm{d}\Omega}{\sqrt{\Omega}} \mathrm{d}U \, , \tag{3.3.19a}$$

$$= 2\pi \int_0^{+\infty} \left[\alpha_{\Omega\omega} \delta(\Omega \mp \Omega') - \beta_{\Omega\omega}^* \delta(\Omega \pm \Omega')\right] \frac{\mathrm{d}\Omega}{\sqrt{\Omega}}, \tag{3.3.19b}$$

$$= \frac{2\pi}{\sqrt{\Omega}} \left[\alpha_{\pm\Omega'\omega} - \beta_{\mp\Omega'\omega}^*\right]. \tag{3.3.19c}$$

Since only values of $\Omega \geq 0$ lead to non-vanishing coefficients, we find that

$$\alpha_{\Omega\omega} = \frac{1}{2\pi} \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{+\infty} e^{-i\omega u + i\Omega U} \, \mathrm{d}U \, , \tag{3.3.20}$$

$$\beta_{\Omega\omega} = \frac{1}{2\pi} \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{+\infty} e^{+i\omega u + i\Omega U} \, \mathrm{d}U \, . \tag{3.3.21}$$

With some manipulation, it can be shown that [see, for example, 38, p. 107]

$$|\alpha_{\Omega\omega}|^2 = e^{\frac{2\pi\Omega}{a}} |\beta_{\Omega\omega}|^2. \tag{3.3.22}$$

Next, we notice that the number of particles seem by the accelerated observer is

$$\langle N_\Omega \rangle = \left\langle b_\Omega^\dagger b_\Omega \right\rangle, \tag{3.3.23a}$$

$$= \int \beta_{\Omega\omega'}^* \beta_{\Omega\omega} \left\langle a_{\omega'} a_\omega^\dagger \right\rangle \mathrm{d}\omega \, \mathrm{d}\omega' \, , \tag{3.3.23b}$$

$$= \int \beta_{\Omega\omega'}^* \beta_{\Omega\omega} \delta(\omega' - \omega) \, \mathrm{d}\omega \, \mathrm{d}\omega' \, , \tag{3.3.23c}$$

$$= \int |\beta_{\Omega\omega}|^2 \, \mathrm{d}\omega \, , \tag{3.3.23d}$$

where we used Eq. (3.3.16) on the previous page. To proceed, we notice that at $\Omega = \Omega'$, the normalization condition Eq. (3.3.17) on the preceding page yields

$$\int_0^{+\infty} |\alpha_{\Omega\omega}|^2 - |\beta_{\Omega\omega}|^2 \, \mathrm{d}\omega = \delta(0), \tag{3.3.24}$$

and Eq. (3.3.22) on the previous page then implies

$$\int_0^{+\infty} |\beta_{\Omega\omega}|^2 \, \mathrm{d}\omega = \frac{\delta(0)}{e^{\frac{2\pi\Omega}{a}} - 1}, \tag{3.3.25}$$

which therefore means

$$\langle N_\Omega \rangle = \frac{\delta(0)}{e^{\frac{2\pi\Omega}{a}} - 1}, \tag{3.3.26}$$

where $\delta(0)$ plays the role of the infinite volume of the spacetime. If we were to pick a particle density rather than a particle number, we would have

$$n_\Omega = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1}, \tag{3.3.27}$$

which can be recognized as the Bose–Einstein distribution at temperature $T = \frac{a}{2\pi}$, showing once again the Unruh effect.

3.4 Criticism of the Fock Space Approach

Fock space representations are extremely useful in order to perform calculations in stationary spacetimes, or in asymptotically stationary spacetimes. Nevertheless, they are restricted by symmetry and by the existence of convenient congruences of observables.

Hence, these representations should mostly be viewed as a calculational tool, rather than a rigorous definition of what QFT actually is. They allow us to do computations conveniently, but do not claim any further advantages.

3.5 Reading Recommendations

Khavkine and Moretti [32] discuss how Fock space representations arise for Gaussian states, and Wald [61] works with them from scratch for stationary spacetimes. Most references on QFTCS will derive the Unruh effect by means of a Bogolyubov transformation, which was originally employed by Hawking [27]. We followed Mukhanov and Winitzki [38], but Wald [60, 61] works more carefully to find the quantum state in the Fock space representation of accelerated observers, rather than only the expectation value.

Four

Path Integrals

4.1 Introduction to Path Integrals

Most introductions to path integrals, such as those found in [42, 62, 66], see path integrals as an alternative formulation of quantum theory and derive it from other principles. In quantum field theory in curved spacetime (QFTCS), this approach fails for not every state admits a path integral formulation. Hence, we shall understand path integrals merely as a notation.

Shortly, the path integral approach is a manner of employing directly the Gell-Mann–Low formula [20]. It states that

$$\langle\Omega|\mathcal{T}[\varphi(x)\cdots\varphi(y)]|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\left\langle 0\middle|\mathcal{T}\left[\varphi(x)\cdots\varphi(y)e^{-i\int_{-T}^{+T} H_I(t)\,\mathrm{d}t}\right]\middle|0\right\rangle}{\left\langle 0\middle|\mathcal{T}\left[e^{-i\int_{-T}^{+T} H_I(t)\,\mathrm{d}t}\right]\middle|0\right\rangle}, \tag{4.1.1}$$

for a small $\epsilon > 0$. In the above, $|\Omega\rangle$ represents the interacting vacuum, while $|0\rangle$ represents the non-interacting vacuum. Hence, it is convenient to write

$$\langle\Omega|\mathcal{T}[\varphi(x)\cdots\varphi(y)]|\Omega\rangle = \int e^{iS[\varphi]}\varphi(x)\cdots\varphi(y)\,\mathcal{D}\varphi\,, \tag{4.1.2}$$

where one "integrates over φ" in order to explicitly impose the Gell-Mann–Low formula from the start.

The Gell-Mann–Low formula holds true for the Minkowski vacuum, but what about other vacua? Notice that the existence of a Hamiltonian requires us to have a stationary spacetime and a stationary vacuum for us to even make sense of the formula. Hence, there is no reason to expect path integrals to make sense in general spacetimes or for general states.

4.2 The Unruh Effect

Our path integral calculation mostly follows the original one due to Unruh and Weiss [57], but Crispino, Higuchi, and Matsas [10, Sec. II.I] also present a summarized version.

Using Euclidean path integrals, we intend to show that the equality*

$$\langle 0_M|\mathcal{T}\varphi(x_1)\cdots\varphi(x_n)|0_M\rangle = \frac{\mathrm{Tr}\left[e^{-\beta H_R}\mathcal{T}\varphi(x_1)\cdots\varphi(x_n)\right]}{\mathrm{Tr}\left[e^{-\beta H_R}\right]} \tag{4.2.1}$$

holds for all events $x_1, \ldots, x_n \in R$ when $\beta = \frac{2\pi}{a}$. In the previous expression, H_R is the Rindler Hamiltonian, *i.e.*, the generator of translations with respect to proper time for an accelerated observer with acceleration a. This is a different way of saying that H_R generates translations with respect to the Rindler coordinate η, or

*Notice that the time ordering operator $\mathcal{T}$ can be regarded as a coordinate-independent object. If $x \in J^+(y)$, this happens in all coordinate systems, and hence the action of $\mathcal{T}$ on $\varphi(x)\varphi(y)$ also does. If x and y are spacelike related, then $\varphi(x)$ and $\varphi(y)$ commute and their ordering is irrelevant.

alternatively that it is a times the boost generator. This equality states that n-point correlation functions in the Minkowski vacuum and in a thermal state for the Rindler Hamiltonian are actually the same, and hence, if it holds, it is a statement of the Unruh effect.

Let us begin by writing H_R down explicitly. The general action for a scalar field in Minkowski spacetime is

$$S[\varphi] = -\int \left[\frac{1}{2}\nabla_a\varphi\nabla^a\varphi + V(\varphi)\right]\sqrt{-g}\,\mathrm{d}^4x\,, \tag{4.2.2}$$

where g is the determinant of the metric. We chose to write the action in this way because it allows us to simply see how it will occur in non-inertial coordinates. Notice also that this time we added a general interaction potential $V(\varphi)$, and hence this derivation is not restricted to free or massless fields.

Using Eq. (2.3.4) on page 13, we can see that the action can be written in Rindler coordinates as

$$S[\varphi] = \int_{r>0}\left[\frac{1}{2(ar)^2}\left(\frac{\partial\varphi}{\partial\eta}\right)^2 - \frac{1}{2}\left(\frac{\partial\varphi}{\partial r}\right)^2 - \frac{(\vec{\boldsymbol{\nabla}}_\perp\varphi)^2}{2} - V(\varphi)\right]ar\,\mathrm{d}\eta\,\mathrm{d}r\,\mathrm{d}^2x_\perp\,, \tag{4.2.3}$$

where we are writing $\vec{\boldsymbol{x}}_\perp = (y, z)$.

There is a difference between Eqs. (4.2.2) and (4.2.3) that should be pointed out. While Eq. (4.2.2) is integrating over modes over the entire Minkowski spacetime, Eq. (4.2.3) is written in a coordinate system that is only defined on the right Rindler wedge. Hence, in writing Eq. (4.2.3), we are already assuming we are only paying attention to what happens on the right Rindler wedge.

In Rindler coordinates, the momentum canonically conjugate to φ is given by

$$\pi = \frac{\delta S}{\delta(\partial_\eta\varphi)} = \frac{\partial_\eta\varphi}{ar} \equiv \frac{1}{ar}\frac{\partial\varphi}{\partial\eta}. \tag{4.2.4}$$

Therefore, we get to the Hamiltonian

$$H_R = \int_{r>0}\left[\frac{\pi^2}{2} + \frac{1}{2}\left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{(\vec{\boldsymbol{\nabla}}_\perp\varphi)^2}{2} + V(\varphi)\right]ar\,\mathrm{d}r\,\mathrm{d}^2x_\perp\,. \tag{4.2.5}$$

We then consider the partition function

$$Z_R(\beta) \equiv \operatorname{Tr}\left[e^{-\beta H_R}\right]. \tag{4.2.6}$$

As a path integral, it can be written as [39, Chap. 71]

$$Z_R(\beta) = \int_{\varphi(0)=\varphi(\beta)} \exp\left(-\int_{r>0}\int_0^\beta ar\left[\frac{\pi^2}{2} + \frac{1}{2}\left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{(\vec{\boldsymbol{\nabla}}_\perp\varphi)^2}{2} + V(\varphi)\right] - i\pi\frac{\partial\varphi}{\partial\tau}\,\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}^2x_\perp\right)\mathcal{D}\varphi\,\mathcal{D}\pi\,, \tag{4.2.7}$$

where $\varphi(0) = \varphi(\beta)$ means the integral runs over field configurations with periodic boundary conditions in τ with period β.

The integral over π is Gaussian. It can be solved by noticing that

$$\frac{ar\pi^2}{2} - i\pi\frac{\partial\varphi}{\partial\tau} = \frac{ar}{2}\left(\pi - \frac{i}{ar}\frac{\partial\varphi}{\partial\tau}\right)^2 + \frac{1}{2ar}\left(\frac{\partial\varphi}{\partial\tau}\right)^2. \tag{4.2.8}$$

Therefore, up to a superfluous normalization factor, one has

$$Z_R(\beta) = \int_{\varphi(0)=\varphi(\beta)} \exp\left(-\int_{r>0}\int_0^\beta \frac{1}{2ar}\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + ar\left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{(\vec{\boldsymbol{\nabla}}_\perp\varphi)^2}{2} + V(\varphi)\right]\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}^2x_\perp\right)\mathcal{D}\varphi\,. \tag{4.2.9}$$

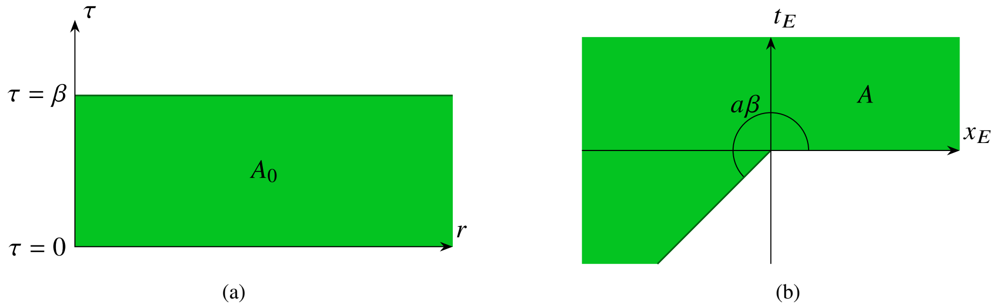

Figure 4.1: Integration region before (A_0) and after (A) the coordinate transformation done on Eq. (4.2.11). Based on Figure 1 of the paper by Unruh and Weiss [57].

Notice this expression can be understood in terms of the Euclidean, finite-temperature version of Eq. (4.2.3) on the facing page. Namely,

$$Z_R(\beta) = \int_{\varphi(0)=\varphi(\beta)} e^{-S^{\beta}_{RE}[\varphi]}\,\mathcal{D}\varphi\,, \tag{4.2.10}$$

where the subscripts "RE" stand for "Rindler" and "Euclidean". Notice that if we had chosen other coordinate systems—such as an inertial coordinate system—the Euclidean action could be different.

We are free to perform the integral in the exponent of Eq. (4.2.9) on the preceding page in whichever way we see fit. In particular, we can perform a change of variables according to

$$t_E = r\sin a\tau \quad \text{and} \quad x_E = r\cos a\tau. \tag{4.2.11}$$

While these are inspired by our definition of Rindler coordinates, notice we are not changing to a new coordinate chart on the manifold. We are only making a change of variables in the integral. One can then show that

$$\begin{aligned}\int_{r>0}\int_0^{\beta} \frac{1}{2ar}\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + ar\left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{(\vec{\nabla}_\perp\varphi)^2}{2} + V(\varphi)\right]\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}^2x_\perp \\ = \int_A \frac{1}{2}\left(\frac{\partial\varphi}{\partial t_E}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x_E}\right)^2 + \frac{1}{2}(\vec{\nabla}_\perp\varphi)^2 + V(\varphi)\,\mathrm{d}t_E\,\mathrm{d}x_E\,\mathrm{d}^2x_\perp\,,\end{aligned} \tag{4.2.12}$$

where the integration region A is illustrated on Fig. 4.1. Notice that Eq. (4.2.11) can only be single-valued if $\beta a \leq 2\pi$.

Consider now the case $\beta = \frac{2\pi}{a}$. We can then write

$$Z_R\left(\frac{2\pi}{a}\right) = \int \exp\left(-\int \frac{1}{2}\left(\vec{\nabla}_4\varphi\right)^2 + V(\varphi)\,\mathrm{d}^4x_E\right)\mathcal{D}\varphi\,, \tag{4.2.13}$$

where we dropped the condition $\varphi(0) = \varphi(\beta)$, because it is now automatically implemented by the new variables. Notice, however, that the right-hand side (RHS) of Eq. (4.2.13) is merely the generating functional at zero source for the theory in inertial coordinates. Hence,

$$Z_R(\beta) = \int e^{-S_{iE}[\varphi]}\,\mathcal{D}\varphi\,, \tag{4.2.14}$$

where "iE" stands for "inertial" and "Euclidean".

These ideas can be generalized in a straightforward manner to a generating functional in the presence of a source, $Z[J]$. In this case, functional derivatives with respect to the source allow us to obtain the n-point correlation functions. We then find that

$$\langle 0_M | \varphi(x_E^1) \cdots \varphi(x_E^n) | 0_M \rangle = \frac{\mathrm{Tr}\left[e^{-\beta H_R} \varphi(x_E^1) \cdots \varphi(x_E^n)\right]}{\mathrm{Tr}\left[e^{-\beta H_R}\right]}. \tag{4.2.15}$$

Notice this is not Eq. (4.2.1) on page 25. Eq. (4.2.15) is an equality among correlation functions on a spacetime of Euclidean signature. However, the RHS of Eq. (4.2.1) on page 25 can be obtained from the RHS of Eq. (4.2.15) by means of the analytic continuation $\tau = i\eta$—this is how we went from Eq. (4.2.6) on page 26 to Eq. (4.2.7) on page 26. Similarly, the left-hand side (LHS) of Eq. (4.2.1) on page 25 can be obtained from the LHS of Eq. (4.2.15) under $t_E = it$. Nevertheless, as one might notice from Eqs. (2.3.3) and (4.2.11) on page 13 and on the preceding page, it turns out that $\tau = i\eta$ and $t_E = it$ are actually the same analytic continuation. Hence, Eq. (4.2.15) implies Eq. (4.2.1) on page 25, concluding our proof.

4.3 Reading Recommendations

The path integral formulation of quantum mechanics and quantum field theory in flat spacetime is well-discussed in the books by Peskin and Schroeder [42] and Zee [66]. For QFTCS, see the book by Mukhanov and Winitzki [38] for an introductory discussion. The master's thesis by the author [1] discusses the path integral formulation at an intermediate level and critiques when it can be used. Although it focuses on the Euclidean signature path integral, Lorentzian path integrals should always be understood as analytic continuations of an Euclidean path integral. Our discussion of the Unruh effect follows the original paper by Unruh and Weiss [57] and the review by Crispino, Higuchi, and Matsas [10].

Five

Particle Detectors

Another interesting approach for deriving the Unruh effect is to employ a particle detector. This allows us to obtain a different point of view on the phenomenom.

5.1 What is a Particle Detector?

We shall consider an Unruh–DeWitt detector [12, 55]. This is a two-level detector that can be excited or de-excited through interactions with the quantum field, similar to how an ammonia molecule can flip states upon interaction with an external electric field [see 16, Chap. 9]. Intuitively, the detector will flip from the ground state to the excited state when it absorbs a "particle", and will decay when it emits a "particle". Pictorially, we are considering a "particle in a box" that can interact with the field. For example, we are carrying around an electron in a box and use it to measure properties of the electromagnetic field. Further details are given by Unruh and Wald [56] and Wald [61, Sec. 3.3]. Our discussion follows the review given by Burbano, Perche, and Torres [8] and also draws from the seminal works by DeWitt [12] and Unruh [55].

We have already discussed at length how to describe a quantum field. For the detector, we shall consider a two-level quantum system—*i.e.*, a qubit— with free Hamiltonian

$$H_\Omega = \frac{\Omega}{2}\sigma^z, \tag{5.1.1}$$

where σ^z is the Pauli matrix and Ω is a constant with dimension of energy. It represents the energy gap between the ground and excited states of the detector. Since we want the excited state to have an energy larger than that of the ground state, we assume $\Omega > 0$. Notice that H_Ω generates translations with respect to the detector's proper time.

We shall also introduce an interaction between the quantum field and the detector. We write, in the interaction picture,

$$H_{\text{int}} = \epsilon\sigma^x(\tau) \otimes \varphi(z(\tau)), \tag{5.1.2}$$

where ϵ is a coupling constant, $\sigma^x(\tau)$ is the Pauli matrix (which evolves in the interaction picture), φ is the quantum field, and $z(\tau)$ denotes the detector's worldline. Hence, we are prescribing a pointlike interaction between detector and field along the detector's worldline. This interaction could be more complex to allow us to turn the detector on and off, or to allow for the detector to have spatial degrees of freedom, but this simple model is sufficient for our present purposes.

The quantum field also evolves with some Hamiltonian H_φ. This Hamiltonian evolves the field along inertial time, and hence we need to introduce a correction factor to account for the evolution with respect to the detector's proper time. This is merely a factor of $\frac{dt}{d\tau}$, since

$$i\frac{d}{d\tau} = i\frac{dt}{d\tau}\frac{d}{dt} = \frac{dt}{d\tau}H_\varphi. \tag{5.1.3}$$

At the end of the day, we have the Hamiltonian

$$H = \frac{dt}{d\tau}H_\varphi + \frac{\Omega}{2}\sigma^z + \epsilon\sigma^x(\tau) \otimes \varphi(z(\tau)). \tag{5.1.4}$$

5.2 Excitation Probability

Let us then compute the excitation probability for the detector. Consider the system's initial state is $|g,0\rangle = |g\rangle \otimes |0\rangle$, where $|g\rangle$ denotes the detector's ground state and $|0\rangle$ denotes the vacuum. We are mainly interested in the Minkowski vacuum, but most of our calculation also works for other states and spacetimes. We want to compute the probability that the system undergoes a transition to some state $|e,\varphi\rangle = |e\rangle \otimes |\varphi\rangle$, where $|e\rangle$ is the detector's excited state and $|\varphi\rangle$ is an arbitrary field state. Hence, we are first trying to compute the amplitude

$$A_{g\to e}(\varphi) = \langle e,\varphi|U_{\text{int}}|g,0\rangle\,, \tag{5.2.1}$$

where U_{int} is the time-evolution operator in the interaction picture.

To compute this expression, we begin by writing the time-evolution operator as a Dyson series [63, Eq. (8.7.13)]

$$\begin{aligned} U_{\text{int}}(\tau',\tau) &= \mathcal{T}\exp\left(-i\int_{\tau}^{\tau'} H_{\text{int}}(\tau_1)\,\mathrm{d}\tau_1\right), && (5.2.2a)\\ &= \mathbb{1} + \sum_{n=1}^{+\infty}\frac{(-i)^n}{n!}\int_{\tau}^{\tau'}\cdots\int_{\tau}^{\tau'}\mathcal{T}(H_{\text{int}}(\tau_1)\cdots H_{\text{int}}(\tau_n))\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n\,, && (5.2.2b)\end{aligned}$$

where $\mathcal{T}$ is the time-ordering operator.

Using Eq. (5.1.2) on the previous page on the Dyson series, we find that

$$U_{\text{int}}(\tau',\tau) = \mathbb{1} + \sum_{n=1}^{+\infty}\frac{(-i\epsilon)^n}{n!}\int_{\tau}^{\tau'}\cdots\int_{\tau}^{\tau'}\mathcal{T}[(\sigma^x(\tau_1)\cdots\sigma^x(\tau_n))\otimes(\varphi(\tau_1)\cdots\varphi(\tau_n))]\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n\,, \tag{5.2.3}$$

where we have now adopted the simplified notation $\varphi(\tau) \equiv \varphi(z(\tau))$.

Let us then consider the amplitude we are interested in. When computing $\langle e,\varphi|U_{\text{int}}|g,0\rangle$, we can immediately see the identity drops out, since the two states are orthogonal. Hence, we are left with

$$\begin{aligned} &A_{g\to e}(\varphi;\tau,\tau')\\ &= \sum_{n=1}^{+\infty}\frac{(-i\epsilon)^n}{n!}\int_{\tau}^{\tau'}\cdots\int_{\tau}^{\tau'}\langle e,\varphi|\mathcal{T}[(\sigma^x(\tau_1)\cdots\sigma^x(\tau_n))\otimes(\varphi(\tau_1)\cdots\varphi(\tau_n))]|g,0\rangle\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n\,. \end{aligned} \tag{5.2.4}$$

Notice that

$$\begin{aligned} &\langle e,\varphi|\mathcal{T}[(\sigma^x(\tau_1)\cdots\sigma^x(\tau_n))\otimes(\varphi(\tau_1)\cdots\varphi(\tau_n))]|g,0\rangle\\ &\qquad = \langle e|\mathcal{T}(\sigma^x(\tau_1)\cdots\sigma^x(\tau_n))|g\rangle\,\langle\varphi|\mathcal{T}(\varphi(\tau_1)\cdots\varphi(\tau_n))|0\rangle\,. \end{aligned} \tag{5.2.5}$$

At this stage, we cannot simplify the n-point function, but we can proceed with our calculation for the detector factor.

Let us begin by noticing that we can write σ^x in terms of ladder operators as

$$\sigma^x = \sigma^+ + \sigma^-, \tag{5.2.6}$$

where

$$\sigma^\pm = \frac{\sigma^x \pm i\sigma^y}{2}. \tag{5.2.7}$$

The advantage of noticing this is that

$$\sigma^+|g\rangle = |e\rangle\,,\quad \sigma^+|e\rangle = 0,\quad \sigma^-|g\rangle = 0,\quad \text{and}\quad \sigma^-|e\rangle = |g\rangle\,, \tag{5.2.8}$$

which imply

$$\sigma^x |g\rangle = |e\rangle \quad \text{and} \quad \sigma^x |e\rangle = |g\rangle . \tag{5.2.9}$$

We then consider the slightly more complicated case where the Pauli matrix is being evolved in time, since we are working in the interaction picture. We then have

$$\begin{aligned}
\sigma^x(\tau)|g\rangle &= \exp(i\tau H_\Omega)\sigma^x \exp(-i\tau H_\Omega)|g\rangle , && (5.2.10a)\\
&= \exp(i\tau H_\Omega)\sigma^x \exp\left(\frac{+i\tau\Omega}{2}\right)|g\rangle , && (5.2.10b)\\
&= \exp\left(\frac{+i\tau\Omega}{2}\right)\exp(i\tau H_\Omega)\sigma^x|g\rangle , && (5.2.10c)\\
&= \exp\left(\frac{+i\tau\Omega}{2}\right)\exp(i\tau H_\Omega)|e\rangle , && (5.2.10d)\\
&= \exp\left(\frac{+i\tau\Omega}{2}\right)\exp\left(\frac{+i\tau\Omega}{2}\right)|e\rangle , && (5.2.10e)\\
&= \exp(+i\tau\Omega)|e\rangle . && (5.2.10f)
\end{aligned}$$

An analogous calculation leads to

$$\sigma^x(\tau)|e\rangle = \exp(-i\tau\Omega)|g\rangle . \tag{5.2.11}$$

Therefore, we find that

$$\langle e|\sigma^x(\tau_1)\cdots\sigma^x(\tau_n)|g\rangle = \begin{cases} e^{i\Omega(\tau_1-\tau_2+\tau_3-\cdots+\tau_n)}, & \text{if } n \text{ is odd,}\\ 0, & \text{if } n \text{ is even.}\end{cases} \tag{5.2.12}$$

In the time-ordered case, we get a similar result, but we must order the terms in the exponential correctly. Hence, we shall simply denote

$$\langle e|\mathcal{T}(\sigma^x(\tau_1)\cdots\sigma^x(\tau_{2n+1}))|g\rangle = \mathcal{T}e^{i\Omega(\tau_1-\tau_2+\tau_3-\cdots+\tau_{2n+1})}, \tag{5.2.13}$$

and the expression vanishes if there is an even number of insertions.

Bringing all of this back to the Dyson series, we find that

$$A_{g\to e}(\varphi;\tau,\tau') = \sum_{n \text{ odd}} \frac{(-i\epsilon)^n}{n!}\int_\tau^{\tau'}\cdots\int_\tau^{\tau'} \langle\varphi|\mathcal{T}(\varphi(\tau_1)\cdots\varphi(\tau_n))|0\rangle\, \mathcal{T}e^{i\Omega(\tau_1-\cdots+\tau_n)}\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n . \tag{5.2.14}$$

The probability for the transition happening between the instants τ and τ' is then

$$p_{g\to e}(\tau,\tau') = \int \left|A_{g\to e}(\varphi;\tau,\tau')\right|^2 \mathcal{D}\varphi , \tag{5.2.15}$$

where we are integrating the field's state out, since we are looking only at the detector. Using the resolution of the identity written as $\int |\varphi\rangle\langle\varphi|\,\mathcal{D}\varphi = \mathbb{1}$, we find that

$$\begin{aligned}
p_{g\to e}(\tau,\tau') = \sum_{n,m \text{ odd}} \epsilon^{n+m}\frac{(-i)^{n-m}}{n!m!}\int_\tau^{\tau'}\cdots\int_\tau^{\tau'} \left\langle 0\middle|\mathcal{T}(\varphi(\tau_1')\cdots\varphi(\tau_m'))^\dagger\mathcal{T}(\varphi(\tau_1)\cdots\varphi(\tau_n))\middle|0\right\rangle \times\\
\times\mathcal{T}e^{i\Omega(\tau_1-\cdots+\tau_n)}\mathcal{T}e^{-i\Omega(\tau_1'-\cdots+\tau_m')}\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n\,\mathrm{d}\tau_1'\cdots\mathrm{d}\tau_m' .
\end{aligned} \tag{5.2.16}$$

Up to leading order, we have

$$p_{g\to e}(\tau,\tau') = \epsilon^2\int_\tau^{\tau'}\int_\tau^{\tau'}\left\langle 0\middle|\varphi(\tau_1')\varphi(\tau_1)\middle|0\right\rangle e^{-i\Omega(\tau_1'-\tau_1)}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_1' . \tag{5.2.17}$$

Notice this expression means the probability of excitation is given by a Fourier transform of the two-point function.

5.3 The Unruh Effect

So far, we did not need to specify the details of the state $|0\rangle$, the worldline of the detector, and not even the spacetime we are working in. This exhibits how useful particle detectors can be in a myriad of situations. Our case of interest concerns the Minkowski vacuum in Minkowski spacetime. For an inertial detector, the proper time τ would coincide with inertial time. In this case, we know the two-point function only has contributions due to positive frequencies (this follows from Eq. (2.3.6) on page 14), and hence the probability will vanish for $\Omega > 0$, which is our case of interest. Hence, an inertial detector will not detect any particles in the Minkowski vacuum, as expected. Nevertheless, notice that accelerated detectors have different frequency decompositions and, as a consequence, may lead to non-vanishing excitation probabilities.

Let us then specify the detector's worldline. This can be done naturally in Rindler coordinates, with which we specify the worldline as

$$z^\mu(\tau) = \left(\tau; \frac{1}{a}, 0, 0\right). \tag{5.3.1}$$

We took $r = \frac{1}{a}$ because this corresponds to the worldline of the observer with proper acceleration a defining Rindler coordinates [44, 45, Sec. 12.4]. We also took the coordinates $y(\tau) = z(\tau) = 0$ for simplicity, but they could have been given any other constant value without altering the following results.

Using Eqs. (2.3.3), (2.3.5) and (5.3.1) on page 13 and on this page, we find that

$$\left\langle 0_M \middle| \varphi(\tau_1')\varphi(\tau_1) \middle| 0_M \right\rangle = \underset{\varepsilon\to 0^+}{\text{w-lim}} \frac{a^2}{4\pi^2\left[(\cosh\left(a\tau_1'\right) - \cosh(a(\tau_1 - i\varepsilon)))^2 - (\sinh\left(a\tau_1'\right) - \sinh(a(\tau_1 - i\varepsilon)))^2\right]}, \tag{5.3.2}$$

where we chose to align the arbitrary future-directed timelike vector T^a of Eq. (2.3.5) on page 13 along the $\left(\frac{\partial}{\partial\eta}\right)^a$ direction, for this simplifies the expression. The previous equation can then be further simplified using the properties of hyperbolic functions to get to

$$\left\langle 0_M \middle| \varphi(\tau_1')\varphi(\tau_1) \middle| 0_M \right\rangle = \underset{\varepsilon\to 0^+}{\text{w-lim}} \frac{-a^2}{16\pi^2 \sinh^2\left(\frac{1}{2}a\left(\tau_1' - \tau_1 - i\varepsilon\right)\right)}. \tag{5.3.3}$$

We can then notice that the probability of excitation is

$$p_{g\to e}(\tau, \tau') = -\frac{a^2\epsilon^2}{16\pi^2} \lim_{\varepsilon\to 0^+} \int_\tau^{\tau'} \int_\tau^{\tau'} \frac{\exp\left(-i\Omega(\tau_1' - \tau_1)\right)}{\sinh^2\left(\frac{a}{2}\left(\tau_1' - \tau_1 - i\varepsilon\right)\right)} \, \mathrm{d}\tau_1' \, \mathrm{d}\tau_1 \,, \tag{5.3.4a}$$

$$= -\frac{a\epsilon^2}{8\pi^2} \lim_{\varepsilon\to 0^+} \int_\tau^{\tau'} \int_{a(\tau-\tau_1)/2}^{a(\tau'-\tau_1)/2} \frac{\exp\left(-\frac{2i\Omega\eta}{a}\right)}{\sinh^2(\eta - i\varepsilon)} \, \mathrm{d}\eta \, \mathrm{d}\tau_1 \,, \tag{5.3.4b}$$

where we defined $\eta = \frac{a(\tau_1' - \tau_1)}{2}$.

Let us then define the rate of excitation through

$$R_{g\to e} = \lim_{\substack{\tau'\to+\infty \\ \tau\to-\infty}} \frac{p_{g\to e}(\tau, \tau')}{\tau' - \tau}, \tag{5.3.5a}$$

$$= -\frac{a\epsilon^2}{8\pi^2} \lim_{\varepsilon\to 0^+} \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{2i\Omega\eta}{a}\right)}{\sinh^2(\eta - i\varepsilon)} \, \mathrm{d}\eta \,. \tag{5.3.5b}$$

This Fourier transform can be computed using the residue theorem. One finds

$$R_{g\to e} = \frac{\epsilon^2\Omega}{2\pi\left(e^{\frac{2\pi\Omega}{a}} - 1\right)}. \tag{5.3.6}$$

If we did the same calculations for the $|e, \varphi\rangle \to |g, 0_M\rangle$ transition, we would get

$$R_{e\to g} = \frac{\epsilon^2 \Omega}{2\pi\left(1 - e^{-\frac{2\pi\Omega}{a}}\right)}, \tag{5.3.7}$$

which is the same result with $\Omega \to -\Omega$. Notice then that this implies

$$\frac{R_{g\to e}}{R_{e\to g}} = e^{-\frac{2\pi\Omega}{a}}, \tag{5.3.8}$$

meaning the detector satisfies the detailed balance [see 54] at inverse temperature $\beta = \frac{2\pi}{a}$. This is a hallmark of a system in thermal equilibrium.

Given how abstract our previous approaches can be, it is interesting to notice how "experimental" this derivation is. While we used a simplified model for a particle detector, many physical systems can be understood as detectors. For example, a thermometer. One can even take it further and understand a steak as a particle detector, in which case the Unruh effect will present itself as a cooking method. Since the Maillard reaction happens at temperatures above 425 K and one needs an acceleration of about $10^{22}\,\mathrm{m\,s^{-2}}$ to reach this Unruh temperature, it might be desirable to sear your steak before cooking it with the Unruh effect.

5.4 Reading Recommendations

The first notions about the Unruh–DeWitt particle detector were laid out by Unruh [55] and by DeWitt [12], with most of our discussion being adapted from the paper by DeWitt [12]. An useful reference to understand how particle detectors work is the paper by Unruh and Wald [56].

A

Notation and Conventions

We follow the notation and conventions used by Wald [59], which corresponds to $+++$ in the Misner, Thorne, and Wheeler [37] classification and employs abstract index notation.

While we often use units with $\hbar = c = G = k_B = 1$, we sometimes write these constants explicitly for clarity.

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