Quantum Information Protocols in QFT

- Our goal: couple local probes to a QFT to implement QI protocols.

This lecture: Background

1) QFT in curved spacetimes
   - ↳ set basics and notation
   - ↳ properties of propagators
2) QI protocols
   - ↳ mixed vs pure states
   - ↳ entanglement
   - ↳ quantum channels → ent. breaking / class. channel cap.
3) Fermi Normal Coordinates
   - ↳ construction
   - ↳ limitations
   - ↳ applications.

QFT in curved Spacetimes

We will focus on the case of a scalar field in curved spacetimes. This can be generalized (see e.g. "Advances in Algebraic Quantum Field Theory" by R. Brunetti, C. Dappiaggi, K. Fredenhagen, J. Yngvason → especially chapter 3)

We will always work with a specific example of AQFT, where the quantum field theory can be built as the following association:

$$f \longmapsto \hat{\Phi}(f), \quad f \in \underbrace{C_0^\infty(M)}_{\to \text{spacetime}} \longrightarrow \text{can be relaxed.}$$

Such that $f \mapsto \hat{\Phi}(f)$ is linear

$$\hat{\Phi}(f)^\dagger = \hat{\Phi}(f^*)$$

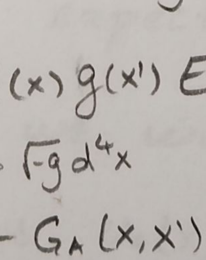
$$\hat{\Phi}(Pf) = 0, \quad P = \text{e.o.m. } \underset{\text{(linear)}}{\text{differential operator}}$$

$$[\hat{\Phi}(f), \hat{\Phi}(g)] = i\, E(f,g),$$

where $E(f,g) = \int_{M\times M'} \underbrace{dV dV'}_{\to \sqrt{-g}\, d^4x} f(x)\, g(x')\, \underbrace{E(x,x')}_{\to \text{causal propagator}}$, and

$$E(x,x') = \underbrace{G_R(x,x')}_{\to \text{retarded green's function}} - \underbrace{G_A(x,x')}_{\to \text{advanced Green's function}}$$

$PG_R = \delta$, $PG_A = \delta$.



Overall, if $A(x,x')$ is a bifunction, we will denote:

$$Af \equiv Af(x) = \int dV' A(x,x') f$$

$$A(f,g) = \int dV dV' f(x) g(x') A(x,x')$$

And overall, the quantum field $\hat{\phi}(x)$ should be thought as the kernel of an operator valued distribution:

$$\hat{\phi}(f) = \int dV f(x) \hat{\phi}(x).$$

The algebras $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \subseteq M$ are then built by products and linear combinations of elements of the form $\hat{\phi}(f)$ s.t. $f \in C_0^\infty(\mathcal{O})$.

$\mathcal{A} \equiv \mathcal{A}(M)$

A state is a linear function $\omega : \mathcal{A} \to \mathbb{C}$ such that $\omega(\mathbb{1}) = 1$, $\omega \geq 0$ $(\omega(A^+A) \geq 0 \; \forall A \in \mathcal{A})$.

(maps operators to expected values)

→ In this course we will rarely pick specific basis of solutions $u_k(x)$. We will not use representations such as

$$\hat{\phi}(x) = \int d^3k \left(u_k(x) \hat{a}_k + u_k^*(x) \hat{a}_k^+ \right)$$

(not necessary)

However, we will mostly restrict ourselves to zero-mean Gaussian states. That is, states such that $\omega(\hat{\Phi}(f_1)\dots\hat{\Phi}(f_{2n+1}))=0$ $\forall n\in\mathbb{N}$ and such that Wick's theorem applies to even products.

Every state defines a correlation function (or Wightman function) through

$$\omega(\hat{\Phi}(f)\hat{\Phi}(g)) = W(f,g) = \int dV dV' f(x) g(x') W(x,x')$$

↳ Wightman function.

An extra restriction to the states that we will consider here is that

$$W(f,g) - W(g,f) = i E(f,g).$$

From the conditions above we have:

$$W(x,x') = \frac{1}{2} H_1(x,x') + \frac{i}{2} E(x,x'),$$

where $H(f,g) = \omega(\{\hat{\Phi}(f),\hat{\Phi}(g)\})$, $H(x,x') = H(x',x) \in \mathbb{R}$ is the Hadamard distribution.

We also define the Feynman propagator:

$$G_F(x,x') = \Theta(t-t')W(x,x') + \Theta(t'-t)W(x',x)$$

for any time parameter t. ↳ Heaviside Θ

Show that:

- G_F is independent of the time parameter t (Hint: $\Theta(t-t')G_A(x,x') = 0$, $\Theta(t-t')G_R(x,x') = G_R(x,x')$)

- $G_F(x,x') = G_F(x',x)$
- $G_F(x,x') = \frac{1}{2}H(x,x') + \frac{i}{2}\Delta(x,x')$, where

$$\Delta(x,x') = G_R(x,x') + G_A(x,x') = \Delta(x',x)$$

(notice that $G_R(x',x) = G_A(x,x')$),

Important properties:

$$\omega(e^{i\hat{\phi}(f)}) = e^{-\frac{1}{2}W(f,f)} = e^{-\frac{1}{4}H(f,f)}$$

$$\overset{*}{W}(x,x') = W(x',x) \iff (W(f,g))^* = W(g^*,f^*)$$

$$E(x,x') = -E(x,x') \iff E(f,g) = -E(g,f)$$

$$G_F(x,x') = G_F(x',x) \iff G_F(f,g) = G_F(g,f)$$

$$\Delta(x,x') = \Delta(x',x) \iff \Delta(f,g) = \Delta(g,f).$$

We will also define the (time parameter dependant) distributions:

$$W_t(x,x') = \Theta(t-t')W(x,x')$$

$$W_{-t}(x,x') = \Theta(t'-t)W(x',x)$$

$$\Rightarrow G_F(x,x') = W_t(x,x') + W_{-t}(x,x')$$

$$\Rightarrow W(x,x') = W_t(x,x') + \overset{*}{W}_{-t}(x,x')$$

because $\Theta(u) + \Theta(-u) = 1$ and $W^*(x',x) = W(x,x')$.

<u>remark</u>: $w^*(f,g) \neq (W(f,g))^* = W(g^*,f^*)$

$\hookrightarrow \int dV dV' f(x) g(x') W^*(x,x') = W(g,f)$

QI Protocols

- States in Quantum theory.

1) $|\psi\rangle \in \mathcal{H}: \langle\psi|\psi\rangle = 1, \quad |\psi\rangle \sim e^{i\phi}|\psi\rangle$
2) $\hat{\rho} \in \mathcal{L}(\mathcal{H}) : \hat{\rho} \geq 0, \ tr(\hat{\rho}) = 1$.
3) $\omega : \cancel{\mathcal{L}(\mathcal{H})} \to \mathbb{C} : \omega(\mathbb{1}) = 1, \ \omega \geq 0.$
   $\mathcal{A}$

Each of these is more general than the other one:

$$\underset{3)}{\hat{A} \in \mathcal{L}(\mathcal{H}) \Rightarrow \omega(\hat{A})} \equiv \underset{2)}{tr(\hat{\rho}\hat{A})}, \quad \hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad \sum p_i = 1, p_i \geq 0$$

$\hookrightarrow$ convex comb.

$$\text{If } \hat{\rho} = |\psi\rangle\langle\psi| \Rightarrow \underset{2)}{tr(\hat{\rho}\hat{A})} = \underset{1)}{\langle\psi|\hat{A}|\psi\rangle}$$

Pure state: ω such that it cannot be written as a convex combination of other states.

- Multi-partite systems

Given two quantum systems represented in $\mathcal{A}_1$ & $\mathcal{A}_2$ or $\mathcal{H}_1$ & $\mathcal{H}_2$, then the composite system is represented in $\mathcal{A}_1 \otimes \mathcal{A}_2$, or $\mathcal{H}_1 \otimes \mathcal{H}_2$.

$\hookrightarrow$ notice that a quantum field here and the same quantum field there are not two systems!

Let's focus on $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $\{|\phi_n\rangle\}$ and $\{|\psi_m\rangle\}$ are bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\{|\phi_n\rangle \otimes |\psi_m\rangle\}$ is a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

However $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \not\Rightarrow |\Psi\rangle = |v_1\rangle \otimes |v_2\rangle$

$\hookrightarrow$ this is the key concept of entanglement.

A pure state which can be written as $|v_i\rangle \otimes |v_i\rangle$ is called a separable state.

For mixed states, a state is separable if it can be written as a convex combination of product states: $\hat{\rho} = \sum_i p_i \, \hat{\rho}_i \otimes \hat{\upsilon}_i$ (for bipartite systems)

A state is said to be entangled if it is not separable.

→ Entanglement allows many different protocol to be performed: secure key distribution, quantum dense cooling, quantum teleportation,...

The key question is how to quantify it in general. → hard!

↳ for pure bipartite states, $S(\hat{\rho}_A) = S(\hat{\rho}_B)$ is an entanglement quantifyer, where

$\hat{\rho}_A = tr_B(|\Psi\rangle\langle\Psi|)$, $\hat{\rho}_B = tr_A(|\Psi\rangle\langle\Psi|)$, $S(\hat{\rho}) = -tr(\hat{\rho} \log \hat{\rho})$

↳ reduced density operator — Von-Newman entropy.

$S(\hat{\rho}) = 0$ for pure states, but $S(\hat{\rho}) \neq 0$ for mixed

It is usually required that an entanglement measure satisfies:

1) $E(\hat{\rho}) = 0$ if $\hat{\rho}$ is separable

2) $E(|\Psi_d\rangle)$ is maximal for $|\Psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle$

3) E is invariant under $U_A \otimes U_B$.

Important for us: <u>Negativity</u>

Peres Criterium: $\hat{\rho}$ state $\Rightarrow \hat{\rho}^t$ state.

If $\hat{\rho} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, $\hat{\rho} = \sum_{ij} c_{ij}\, \hat{\rho}_i \otimes \hat{\sigma}_j$. Define the partial transpose: $\hat{\rho}^{t_B} = \sum_{ij} c_{ij}\, \hat{\rho}_i \otimes \hat{\sigma}_j^{\,t}$, and it might be that $\hat{\rho}^{t_B}$ is not positive.

$$\boxed{\hat{\rho}^{t_B} \text{ not positive} \Rightarrow \hat{\rho} \text{ entangled}}$$

↳ for 2 qubits $\Leftrightarrow$

<u>Negativity</u>: $N(\hat{\rho}) = \sum_{\lambda_i \in \Lambda_\rho^-} |\lambda_i|$, $\Lambda_\rho^- = \{\lambda : \lambda < 0,\ \lambda \in \mathrm{spec}(\hat{\rho}^{t_B})\}$.

↳ we will use this entanglement quantifyer a lot (but there are other options on the market).

<u>Quantum Channels</u>

A quantum channel is an operation $\mathcal{E} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ which maps density operators in a given space to density operators in (possibly another) space.

e.g. $\mathcal{E}(\hat{\rho}) = U \hat{\rho} U^\dagger$, $\mathcal{E}_{\hat{\rho}_B}(\hat{\rho}_A) = \mathrm{tr}_A(U \rho_A \otimes \rho_B U^\dagger)$.

For an operation $\mathcal{E} : \mathcal{L}(\mathcal{H}_1) \longrightarrow \mathcal{L}(\mathcal{H}_2)$ to be <u>valid</u> it needs to be trace preserving and completely positive (CPTP):

$$\underset{(TP)}{\mathrm{tr}_2(\mathcal{E}(\hat{\rho})) = \mathrm{tr}_1(\hat{\rho})}, \qquad \underset{(CP)}{\hat{\sigma} \otimes \mathcal{E}(\hat{\rho}) \geq 0 \ \forall\, \hat{\sigma}, \hat{\rho} \geq 0.}$$

We will be interested in two main properties of quantum channels $\mathcal{E}: \mathcal{L}(\mathcal{H}_A) \longrightarrow \mathcal{L}(\mathcal{H}_B)$

Classical Channel Capacity: $C(\mathcal{E})$,
Quantum Channel Capacity: $Q(\mathcal{E})$

These quantify the amount of classical/quantum information that can be transmitted through this quantum channel.

$C(\mathcal{E})$: Alice has a message $m \in X$, X = possible messages, that she wants to transmit to Bob using the channel $\mathcal{E}$ multiple times.

$$m \underset{\text{encodes message}}{\longmapsto} \hat{\rho}_m^{(n)} \in \mathcal{L}(\mathcal{H}_A^{\otimes n}) \xrightarrow[\text{sends to Bob}]{} \mathcal{E}^{\otimes n}(\hat{\rho}_m^{(n)}) \in \mathcal{L}(\mathcal{H}_B^{\otimes n})$$

Bob will now apply a POVM $\{\hat{E}_{m'}\}$ (→ in $\mathcal{H}_B^{\otimes n}$), so that each outcome is associated to a message. The probability that the decoded message is the sent message is:

$$P_{m'=m} = \mathrm{tr}\left(\hat{E}_m \, \mathcal{E}^{\otimes n}(\hat{\rho}_m^{(n)})\right).$$

The rate of communication between Alice and Bob in this protocol is $R_c = \frac{\log_2(|X|)}{n}$ (bits per use of the channel) and it has maximum error $P_{err} = \max_{m \in X}(1 - P_{m'=m})$.

"The classical channel capacity is the supremum over the R_c's with smallest error". → hard to compute
↳ requires optimizing a protocol for the channel.

Luckly, the HSW theorem states that

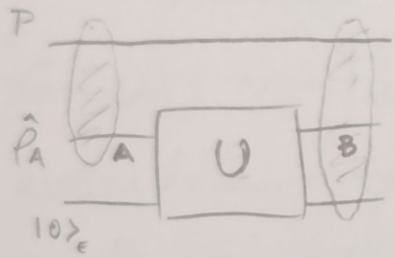
$$C(\mathcal{E}) = \lim_{n\to\infty} \frac{\chi(\mathcal{E}^{\otimes n})}{n},$$

where χ is the Holevo information of $\mathcal{E}^{\otimes n}$:

$$\chi(\mathcal{E}) = \max_{\{p_m, \hat{\rho}_m\}} \left(S(\mathcal{E}(\hat{\rho})) - \sum_m p_m S(\mathcal{E}(\hat{\rho}_m)) \right)$$

<u>$Q(\mathcal{E})$</u>

The main differences from classical and quantum information arise from correlations that systems can have with other systems.



$\mathcal{E}(\hat{\rho}_A) = tr_E(U \hat{\rho}_A \otimes |0\rangle\langle 0|_E U^\dagger)$

$\hat{\rho}_E = tr_B(\hat{U}\, \hat{\rho}_A \otimes |0\rangle\langle 0|_E U^\dagger)$

what if we apply $\mathbb{1} \otimes \mathcal{E}(\hat{\rho}_{PA})$? What correlations do we break?

→ this is very very hard to compute.

→ for us, what matters is that if $\mathbb{1} \otimes \mathcal{E}(\hat{\rho}_{PA})$ is always separable, then $Q(\mathcal{E}) = 0$.

<u>Entanglement Breaking channels</u>:

$(\mathbb{1} \otimes \mathcal{E})(\hat{\rho}_{PA})$ always separable. $\Rightarrow Q(\mathcal{E}) = 0.$

$\Rightarrow C(\mathcal{E}) = \chi(\mathcal{E}). \quad (\chi(\mathcal{E}^{\otimes n}) = n\chi(\mathcal{E})).$

Fermi Normal Coordinates

arXiv: 1102.0329

Let $z(\tau)$ be a timelike curve parametrized by proper time, and denote its four-velocity by u^μ. 1) Pick τ_0, and define $e_0(\tau_0) = u(\tau_0)$. Pick vectors $e_i(\tau_0) \in T_{z(\tau_0)}M$ such that

$$g(e_i, e_j) = \delta_{ij}, \quad g(u, e_i) = 0$$

This defines an orthonormal frame $e_\mu(\tau_0)$ at $T_{z(\tau_0)}M$.

Fermi Transport

Given $v \in T_{z(\tau_0)}M$, its Fermi transport is the solution to

$$\frac{Dv^\mu}{d\tau} + 2a^{[\mu}u^{\nu]}v_\nu = 0, \qquad a^\mu = \frac{Du^\mu}{d\tau}$$

$\rightarrow \frac{D}{d\tau} = u^\alpha \nabla_\alpha$.

$\hookrightarrow$ 4-acceleration

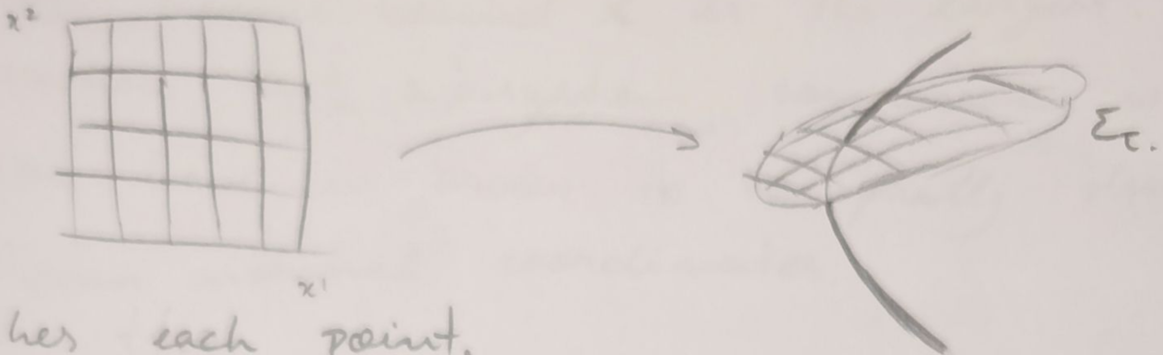
Notice: $u^\mu(\tau)$ is Fermi transported.

Notice: Fermi transport preserves inner products.

2) Extend the frame $\{e_\mu(\tau_0)\}$ to the whole curve via the Fermi transport. $\Rightarrow \{e_\mu(\tau)\}$.

3) For each τ, consider the geodesics that start at $z(\tau)$ with initial velocity $x^i e_i(\tau)$. Follow this geodesic for a proper distance of $\sqrt{\delta_{ij} x^i x^j}$.

This defines a set $\Sigma_\tau \subseteq M$, the rest space of $z(\tau)$, where there is a unique geodesic that rea-



ches each point.

4) repeat this procedure for each τ. Now $(\tau, x^1, \dots, x^n)$ define coordinates in a worldtube around $z(\tau)$

<u>Important:</u> ∂_τ is generically not normal to Σ_τ, and it is usually not normalized.

5)* Parallel transport $\{e_\mu(\tau)\}$ along each geodesic shot in step 3), so that we have a local frame defined in the worldtube.

<u>Useful</u> <u>property</u>:

$$g_{\tau\tau} = -(1 + \underbrace{a_i}_{a_i(\tau)} x^i)^2 - \underbrace{R_{0i0j}}_{R_{0i0j}(z(\tau))} x^i x^j + O(|x|^3)$$

also $O(R^2)$ and $O(a^2)$

$$g_{\tau i} = -\frac{2}{3} R_{0jik} x^j x^k + O(|x|^3)$$

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l + O(|x|^3)$$

The Fermi Bound arXiv: 2206.01225.

The Fermi bound l is the largest proper radius that a system ("rigid") comoving with $z(\tau)$ can have in order to be fully described in Fermi normal coordinates.

Define the τ-Fermi bound l_τ as:

$$l_\tau = \sup\left(\left\{\sqrt{\delta_{ij} x^i x^j} : \exp_{z(\tau)}(x^i e_i(\tau)) \subseteq \Sigma_\tau\right\}\right)$$

$\Rightarrow$ largest radius a system can have in order to be contained in Σ_τ.

The Fermi radius is

$$l = \inf_\tau l_\tau$$

<u>Important:</u> $l \gtrsim \inf_\tau \left(\dfrac{1}{a(\tau) + \sqrt{|\lambda_R(\tau)|}}\right)$,

where $a(\tau)^2 = a_\mu(\tau) a^\mu(\tau)$, $\lambda_R(\tau)$ is the most negative eigenvalue of $R_{0i0j}(z(\tau))$.

Basically: can only use F.N.C. to describe systems that are centered at $z(\tau)$ and have radius smaller than l.