Quantum Information Protocols in QFT

- Our goal: couple local probes to a QFT to implement QI protocols.

Local Probes

- UDW detectors / Particle Detector Models
- Ingredients:

1 QFT + 1 Localized quantum system + 1 interaction.
↳ described in spacetime

QFT

→ Quantum Field in Curved Spacetimes
$(\mathcal{A}, \mathcal{M}, g)$, $\hat{\phi}(f)$, $\omega \longrightarrow W(x, x')$.

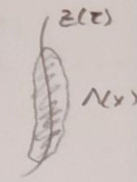
→ Can be a general QFT as well.

Localized Quantum System

→ Quantum degree of freedom localized in a timelike trajectory $z(\tau)$.
↳ proper time
↳ Hilbert space $\mathcal{H}_D$.

→ Free Hamiltonian $\hat{H}_D$ with (at least some) discrete energy levels.
↳ $\hat{H}_D$ promotes time evolution with respect to the proper time τ.

Interaction



- $\rightarrow$ spacetime interaction region $\Lambda(x)$
  - $\hookrightarrow$ spacetime smearing function.
- $\rightarrow$ Field observable $\hat{O}(x)$ (e.g. $\hat{\phi}(f)$, $:\hat{\phi}(f)^2:$, ...)
- $\rightarrow$ Detector observable $\hat{\mu} : \mathcal{H}_0 \longrightarrow \mathcal{H}_0$
- $\rightarrow$ Interaction strength λ.

Result: Interaction Hamiltonian Density:

$$\hat{h}_I(x) = \lambda \Lambda(x) \hat{\mu}(\tau) \hat{\Phi}(x)$$

$\hookrightarrow$ interaction picture

$$\hat{H}_I(\tau) = \int_{\Sigma_\tau} d^3x \sqrt{-g}\, \hat{h}_I(x)$$

Obs: $\tau = \tau(x)$ is the Fermi normal coordinate time $\rightarrow$ more about this later

What can you do with this model?

ex: $\mathcal{H}_D = \mathbb{C}^2$, $\hat{H}_D = \Omega \sigma^+ \sigma^-$ $\rightarrow$ defines $\{|g\rangle, |e\rangle\}$ (ground / excited)

$$\hat{\mu}(\tau) = \hat{\sigma}_x(\tau) = e^{i\Omega\tau} \hat{\sigma}^+ + e^{-i\Omega\tau} \hat{\sigma}^-$$

$$\Rightarrow \hat{h}_I(x) = \lambda \Lambda(x) \left(e^{i\Omega\tau} \hat{\sigma}^+ + e^{-i\Omega\tau} \hat{\sigma}^- \right) \hat{\phi}(x)$$

two-level UDW model.

Time evolution:

$$\hat{U}_I = T_\tau \exp\left(-i \int dV\, \hat{h}_I(x)\right), \quad dV = d^4x \sqrt{-g}$$

$$= \mathbb{1} \underbrace{- i \int dV\, \hat{h}_I(x)}_{\hat{U}_I^{(1)}} \underbrace{- \int dV dV' \Theta(\tau - \tau') \hat{h}_I(x) \hat{h}_I(x')}_{\hat{U}_I^{(2)}} + \mathcal{O}(\lambda^3)$$

$\rightarrow$ small λ

(Dyson Series)

$$\Rightarrow \hat{\rho}_t = \hat{\rho}_0 + \hat{U}_I^{(1)}\hat{\rho}_0 + \hat{\rho}_0\hat{U}_I^{(1)\dagger} + \hat{U}_I^{(2)}\hat{\rho}_0 + \hat{U}_I^{(1)}\hat{\rho}_0\hat{U}_I^{(1)\dagger} + \hat{\rho}_0\hat{U}_I^{(2)} + \dots$$

$$\Rightarrow \hat{\rho}_D = \mathrm{tr}_\phi(\hat{\rho}_t)$$

$$= \hat{\rho}_{0,D} + 0 + 0 + \mathrm{tr}_\phi\left(\hat{U}^{(2)}\hat{\rho}_0 + \hat{U}_I^{(1)}\hat{\rho}_0\hat{U}_I^{(1)\dagger} + \hat{\rho}_0\hat{U}_I^{(2)}\right) + \dots$$

Say that $\hat{\rho}_0 = \hat{\rho}_{0,D} \otimes \hat{\rho}_\omega \longrightarrow$ rep. of state ω.

$$\Rightarrow \mathrm{tr}_\phi(\hat{U}_I^{(2)}\hat{\rho}_0) = -\int dV dV' \,\Theta(\tau-\tau')\,\mathrm{tr}_\phi(\hat{h}_I(x)\, h_I(x')\,\hat{\rho}_\omega)\,\hat{\rho}_{0,D}$$

$$= -\lambda^2 \int dV dV' \,\Theta(\tau-\tau')\,\Lambda(x)\Lambda(x')\,W(x,x')\,\hat{\mu}(\tau)\hat{\mu}(\tau')\,\hat{\rho}_{0,D}$$

$$e^{i\Omega(\tau-\tau')}\hat{\sigma}^+\hat{\sigma}^- + e^{-i\Omega(\tau-\tau')}\hat{\sigma}^-\hat{\sigma}^+$$

$$= -\lambda^2\left(W_\tau(\Lambda^+,\Lambda^-)\hat{\sigma}^+\hat{\sigma}^- + W_\tau(\Lambda^-,\Lambda^+)\hat{\sigma}^-\hat{\sigma}^+\right)\hat{\rho}_{0,D}$$

$$\Lambda^\pm(x) = e^{\pm i\Omega\tau}\Lambda(x)$$

$$\Rightarrow \mathrm{tr}_\phi(\hat{U}_I^{(1)}\hat{\rho}_0\hat{U}_I^{(1)\dagger}) = \int dV dV' \,\mathrm{tr}_\phi(\hat{h}_I(x)\,\hat{\rho}_0\,\hat{h}_I(x'))$$

$$= \lambda^2 \int dV dV' \,\Lambda(x)\Lambda(x')\,\underbrace{W(x',x)}_{\mathrm{tr}_\phi(\hat{\phi}(x)\,\hat{\rho}_\omega\,\hat{\phi}(x'))}\,\hat{\mu}(\tau)\,\hat{\rho}_{0,D}\,\hat{\mu}(\tau')$$

$$= \lambda^2\left(W(\Lambda^+,\Lambda^+)\hat{\sigma}^+\hat{\rho}_{0,D}\hat{\sigma}^+ + W(\Lambda^-,\Lambda^+)\hat{\sigma}^+\hat{\rho}_{0,D}\hat{\sigma}^- + W(\Lambda^+,\Lambda^-)\hat{\sigma}^-\hat{\rho}_{0,D}\hat{\sigma}^+ + W(\Lambda^-,\Lambda^-)\hat{\sigma}^-\hat{\rho}_{0,D}\hat{\sigma}^-\right)$$

$W_\tau(x,x') = \Theta(\tau-\tau')\,W(x,x')$

$W_{-\tau}(x,x') = \Theta(\tau'-\tau)\,W(x',x)$

$(W_\tau(f,g))^*$

$= \int dV dV' \,\Theta(\tau-\tau') f(x) g(x') W(x,x'$

$= \int dV dV' \,\Theta(\tau-\tau') f^*(x) g^*(x') W(x',$

$= W_{-\tau}^*(g^*, f^*)$

$$\mathrm{tr}_\phi(\hat{\rho}_0 U_I^{(2)\dagger}) = \left(\mathrm{tr}_\phi(\hat{U}_I^{(2)\dagger}\hat{\rho}_0)\right)^\dagger$$

$$\longrightarrow (W_\tau(f,g))^* = W_{-\tau}^*(g^*, f^*)$$

$$= -\lambda^2\hat{\rho}_0\left((W_\tau(\Lambda^+,\Lambda^-))^*\hat{\sigma}^-\hat{\sigma}^+ + (W_\tau(\Lambda^-,\Lambda^+))^*\hat{\sigma}^+\hat{\sigma}^-\right)$$

Putting everything together, we find a quantum channel $\hat{\rho}_{0,D} \longmapsto \hat{\rho}_D$ to leading order in λ.

Specifically, if $\hat{\rho}_{0,D} = |g\rangle\langle g| = \hat{\sigma}^-\hat{\sigma}^+$, we find:

$$\hat{\rho}_D = \hat{\sigma}^-\hat{\sigma}^+ - \lambda^2 W_\tau(\Lambda^-,\Lambda^+)\hat{\sigma}^-\hat{\sigma}^+ + \lambda^2 W(\Lambda^-,\Lambda^+)\hat{\sigma}^+\hat{\sigma}^- - \lambda^2 W_\tau^*(\Lambda^-,\Lambda^+)\hat{\sigma}^-\hat{\sigma}^+$$

$$\Rightarrow \hat{\rho}_D = (1 - \lambda^2 W(\Lambda^-, \Lambda^+))\hat{\sigma}^-\hat{\sigma}^+ + \lambda^2 W(\Lambda^-, \Lambda^+)\hat{\sigma}^+\hat{\sigma}^-$$

$$:= \begin{pmatrix} 1-\mathcal{L} & \\ & \mathcal{L} \end{pmatrix}, \quad \text{where} \quad \mathcal{L} = \lambda^2 W(\Lambda^-, \Lambda^+)$$

Notice: this is a mixed state

$\mathcal{L}$ is the excitation probability. (for $\Omega > 0$)

e.g. Minkowski spacetime, $\Box\phi = m^2\phi$, $\omega \longleftrightarrow |0\rangle$, $z(t) = (t, \vec{x}_0)$

$\Lambda(x) = \chi(t) f(\vec{x})$ $\quad \hookrightarrow W(x,x') = \frac{1}{2\pi^3}\int \frac{d^3k}{2\omega_k} e^{-i\omega_k(t-t')} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$, $\quad \omega_k \to \sqrt{m^2+k^2}$

χ: $\hookrightarrow$ real switching; f: $\hookrightarrow$ real smearing

$$\Rightarrow \mathcal{L} = \lambda^2 \int dt\,dt'\,d^3x\,d^3x'\,\chi(t)e^{-i\Omega t}\chi(t')e^{i\Omega t'} f(\vec{x})f(\vec{x}')$$

$$\times \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega_k} e^{-i\omega_k(t-t')} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

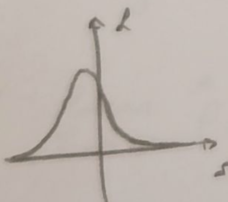
$\tilde{\chi}(\omega) \equiv \int dt\,\chi(t) e^{-i\omega t}$

$$= \frac{\lambda^2}{(2\pi)^3}\int \frac{d^3k}{2\omega_k} |\tilde{\chi}(\omega_k+\Omega)|^2 |\tilde{f}(\vec{k})|^2$$

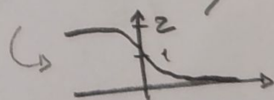
For concreteness, $\chi(t) = e^{-\frac{\pi t^2}{2T^2}}$, $f(\vec{x}) = \frac{e^{-\frac{|\vec{x}|^2}{2\sigma^2}}}{(2\pi\sigma^2)^{3/2}}$, $m = 0$

$$\Rightarrow \tilde{\chi}(\omega) = \sqrt{2\pi}T\, e^{-\frac{\omega^2T^2}{2}}, \quad \tilde{f}(\vec{k}) = e^{-\frac{k^2\sigma^2}{2}}$$

$$\Rightarrow \mathcal{L} = \frac{\lambda^2T^2}{(2\pi)^3}\int \frac{d^3k\,\pi e^{-(k+\Omega)^2T^2 - k^2\sigma^2}}{|k|} = \frac{\lambda^2T^2}{4\pi}\int_0^\infty dk\, k\, e^{-(k+\Omega)^2T^2} e^{-\sigma^2k^2}$$



$$= \frac{\lambda^2}{4\pi}\frac{e^{-\Omega^2T^2}}{1+\sigma^2/T^2}\left(1 - \frac{\sqrt{\pi}\,\Omega T}{\sqrt{1+\sigma^2/T^2}} e^{\frac{\Omega^2T^2}{1+\sigma^2/T^2}} \operatorname{erfc}\left(\frac{\Omega T}{\sqrt{1+\sigma^2/T^2}}\right)\right)$$

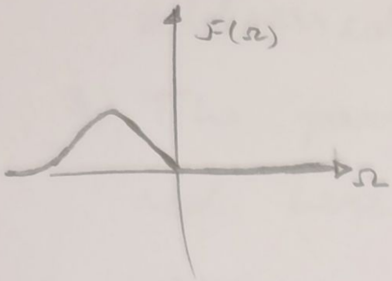


$$\lim_{\sigma\to 0}\mathcal{L} = \frac{\lambda^2}{4\pi}\left(e^{-\Omega^2T^2} - \sqrt{\pi}\,\Omega T \operatorname{erfc}(\Omega T)\right)$$

$$\lim_{T\to\infty}\mathcal{L} = \begin{cases} 0, & \Omega > 0 \\ \infty, & \Omega < 0 \rightarrow \text{perturbation theory fails!} \end{cases}$$

Excitation rate:

$$\mathcal{F}(\Omega) = \lim_{T\to\infty} \frac{\mathcal{L}}{\lambda^2 T} = -\frac{\Omega}{2\pi} e^{-\Omega^2\sigma^2} \Theta(-\Omega) \cdot \sqrt{\pi}$$



↳ transition per unit time

⇒ only deexcitations!

Note: for $T < \infty$, there is a non-zero vacuum probability. → vacuum fluctuations
→ energy comes from switching the interaction on/off.

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Assume: $W(z(\tau), z(\tau')) = W(\tau - \tau')$ (stationary traj.)
$\chi(\tau)$ adiabatically switched with T.
Markovian regime for the interaction.

Let $\alpha = \dfrac{1}{1 + \mathcal{F}(-\Omega)/\mathcal{F}(\Omega)}$, then $\lim_{T\to\infty} \hat{\rho}_D = \begin{pmatrix} 1-\alpha & \\ & \alpha \end{pmatrix}$
(non-perturbatively)
for any choice of initial state.

e.g. $z(\tau) = (\frac{1}{a}\sinh(a\tau), \frac{1}{a}\cosh(a\tau), 0, 0)$ in Minkowski gives

$$\lim_{T\to\infty} \hat{\rho}_D = \frac{1}{1 + e^{-\frac{2\pi}{a}\Omega}} \begin{pmatrix} 1 & \\ & e^{-\frac{2\pi}{a}\Omega} \end{pmatrix} = \frac{e^{-\beta\hat{H}}}{\mathrm{tr}(\beta\hat{H})}, \quad \beta = \frac{2\pi}{a}.$$

- Comments:

1) These detector models seem like complete abstractions, but they do correspond to physical systems.

2) There are important comments about causality, and relativity that have to be addressed.

3) The particle detector models that we discussed here can be generalized in many ways.

4) The choice of time function $\tau = \tau(x)$ has to be physically justified.

→ We will talk about these points in Lecture 4.

Special cases:

Case 1: $\Omega = 0$

In this case we can use the magnus expansion:

$$T\exp\left(\int_{-\infty}^{t} dt_1\, \hat{A}(t_1)\right) = e^{\sum_{n=1}^{\infty} \hat{B}_n(t)}, \quad \text{where}$$

$$\hat{B}_1(t) = \int_{-\infty}^{t} \hat{A}(t_1)\, dt, \quad \hat{B}_2(t) = \frac{1}{2}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\, [\hat{A}(t_1), \hat{A}(t_2)]$$

$$\hat{B}_3(t) = \frac{1}{6}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3 \left([\hat{A}(t_1), [\hat{A}(t_2), \hat{A}(t_3)]] + [\hat{A}(t_3), [\hat{A}(t_2), \hat{A}(t_1)]]\right)$$

$\hat{B}_4(t) =$ 4 commutators of $\hat{A}$, $\hat{B}_5(t) =$ 5 comm...

So we can write

$$\int d^3x \sqrt{-g}\, \hat{h}_I(x)$$

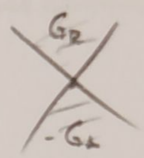
$$T\exp\left(-i\int_{-\infty}^{\infty} d\tau\, \hat{H}_I(\tau)\right) = e^{\sum_{n=1}^{\infty} \Theta_n}, \quad \text{where}$$

$$\hat{\Theta}_1 = -i\int dV\, \hat{h}_I(x) = -i\lambda\hat{\mu}\hat{\phi}(\Lambda)$$

$$\hat{\Theta}_2 = \frac{1}{2}\int dV dV' \,[\hat{h}_I(x), \hat{h}_I(x')]\,\Theta(\tau-\tau')$$

$$= -\frac{\lambda^2}{2}\underbrace{\hat{\mu}^2}_{\mathbb{1}}\int dV dV'\,[\hat{\phi}(x),\hat{\phi}(x')]\,\Lambda(x)\Lambda(x')\,\Theta(\tau-\tau')$$

Remember: $[\hat{\phi}(x),\hat{\phi}(x')] = iE(x,x') = i\left(G_R(x,x') - G_A(x,x')\right)$



$$\Rightarrow \hat{\Theta}_2 = -i\frac{\lambda^2}{2}G_R(\Lambda,\Lambda)\,\mathbb{1} \propto \mathbb{1} \Rightarrow \hat{\Theta}_n = 0 \;\; \forall n \geq 3$$

$$\Rightarrow \hat{U}_I = e^{-i\lambda\hat{\mu}\hat{\phi}(\Lambda) - i\underbrace{\frac{\lambda^2}{2}G_R(\Lambda,\Lambda)}_{\Delta}} = e^{-i\Delta}\, e^{-i\lambda\hat{\mu}\hat{\phi}(\Lambda)}$$

Now, $e^{-i\lambda\hat{\mu}\hat{\phi}(\Lambda)} = \sum_{n=0}^{\infty}\frac{(-i\lambda)^{2n}}{(2n)!}\underbrace{\hat{\mu}^{2n}}_{1}(\hat{\phi}(\Lambda))^{2n} + \sum_{n=0}^{\infty}\frac{(-i\lambda)^{2n+1}}{(2n+1)!}\underbrace{\hat{\mu}^{2n+1}}_{\hat{\mu}}(\hat{\phi}(\Lambda))^{2n+1}$

$$= \cos(\lambda\hat{\phi}(\Lambda)) - i\hat{\mu}\sin(\lambda\hat{\phi}(\lambda))$$

$$\Rightarrow \hat{\rho}_f = \hat{\rho}_{D,0}\cos(\lambda\hat{\phi}(\Lambda))\,\hat{\rho}_\omega\cos(\lambda\hat{\phi}(\Lambda))$$
$$+ i\hat{\rho}_{D,0}\hat{\mu}\cos(\lambda\hat{\phi}(\Lambda))\,\hat{\rho}_\omega\sin(\lambda\hat{\phi}(\Lambda))$$
$$- i\hat{\mu}\hat{\rho}_{D,0}\sin(\lambda\hat{\phi}(\Lambda))\,\hat{\rho}_\omega\cos(\lambda\hat{\phi}(\Lambda))$$
$$+ \hat{\mu}\hat{\rho}_{D,0}\hat{\mu}\sin(\lambda\hat{\phi}(\Lambda))\sin(\lambda\hat{\phi}(\Lambda))$$

$$\Rightarrow \hat{\rho}_D = \mathrm{tr}_\phi(\hat{\rho}_f) = \omega(\cos^2(\lambda\hat{\phi}(\Lambda)))\,\hat{\rho}_{D,0} + \omega(\sin^2(\lambda\hat{\phi}(\Lambda)))\,\hat{\mu}\hat{\rho}_{D,0}\hat{\mu}$$

$$= \frac{1+e^{-2\lambda^2 W(\Lambda,\Lambda)}}{2}\,\hat{\rho}_{D,0} + \frac{1-e^{-2\lambda^2 W(\Lambda,\Lambda)}}{2}\,\hat{\mu}\hat{\rho}_{D,0}\hat{\mu}$$

$$\Rightarrow \hat{\rho}_D = e^{-\xi}\left(\cosh(\xi)\,\hat{\rho}_{D,0} + \sinh(\xi)\,\hat{\mu}\,\hat{\rho}_{D,0}\,\hat{\mu}\right),$$

$$\xi = \lambda^2 W(\Lambda, \Lambda).$$

Case 2: $\Lambda(x) = \underbrace{\eta}_{\text{dimensions of time}}\, \delta(\tau - \tau_0) f(\vec{x})$ (Delta coupling)

In this case, the time ordering is not necessary, and

$$\hat{U}_I = \exp\left(-i \int dV\, \hat{h}_I(x)\right)$$

$$= e^{-i\lambda\eta\,\hat{\mu}(\tau_0)\,\hat{\phi}(f)} \longrightarrow \hat{\phi}(f) = \int d^3x \sqrt{-g}\, f(\vec{x})\, \hat{\phi}(\tau_0, \vec{x})$$

Out of similarity with the previous case, we can conclude that:

$$\hat{\rho}_D = e^{-\xi}\left(\cosh(\xi)\,\hat{\rho}_{D,0} + \sinh(\xi)\,\hat{\mu}(\tau_0)\,\hat{\rho}_{D,0}\,\hat{\mu}(\tau_0)\right),$$

this time with $\xi = \lambda^2 \eta^2 W(f, f)$

$$\hookrightarrow \int d^3x\, d^3x' \sqrt{-g}\sqrt{-g'}\, f(\vec{x}) f(\vec{x}')\, W(\tau_0, \vec{x}; \tau_0, \vec{x}')$$

Importantly, the parameter ξ contains vital information about the quantum field within the support of Λ.

$\hookrightarrow$ One can infer properties about the field by having access to ξ alone!